Mathematical Tables
and other
Aids to Computation

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by

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Admiralty Computing Service[1]

In 1942, in order to use more efficiently the scientific staff available in the Admiralty, the Director of Scientific Research set up, within the branch directed by Dr. J. A. CARROLL, an Admiralty Computing Service to centralise, where possible, the computational and mathematical work arising in Admiralty Experimental Establishments.

Mr. JOHN TODD undertook the organisation and supervision of the Service. By agreement with the Astronomer Royal additional staff were attached to H. M. Nautical Almanac Office to carry out the computational work under the direction of the Superintendent, Mr. D. H. SADLER. In addition, arrangements were made to permit the employment of experts from the Universities and elsewhere as consultants.

The work undertaken by Admiralty Computing Service was in general of one of two classes: heavy computation, or difficult mathematics. Altogether more than one hundred separate investigations were carried out ranging from projects involving several thousand hours' computing to small problems for which a solution could be obtained in a few hours. In addition, a considerable amount of advisory work has been undertaken, usually informally. For instance the *Five-figure Logarithm Tables*, reviewed in RMT **188**, were designed by Admiralty Computing Service for the Ministry of Supply, as one item in a comprehensive program for providing the optical industry with the tables they required. For various reasons, mainly owing to the increased use being made of machines and to the availability of the U. S. reprint of PETERS' seven-figure table of natural trigonometric functions, other tables were never published; but see under UMT **57**.

Shortly after the formation of Admiralty Computing Service it became apparent that research work in Admiralty (and other) Establishments would be greatly facilitated if their members were informed in certain mathematical and computational techniques not usually covered in undergraduate courses, and of which no adequate account was available in easily accessible literature. Accordingly the preparation of a series of monographs of an expository nature was begun [see ACS **53**, **68**, **71**, **101**, **102**, **106** (revised edition of ACS **26**), **107**].

It was soon realised that while centralisation within a Department was an improvement, nothing less than centralisation on a national scale could be really efficient. At the end of 1943, an approach was therefore made to Sir EDWARD V. APPLETON, Secretary of the Department of Scientific and Industrial Research, asking for consideration of the formation of a National Mathematical Laboratory. Discussions, in which the experience gained by Admiralty Computing Service played an important part, have now resulted in the formation of a Mathematics Division of the National Physical Laboratory. Staff have been released from Admiralty Computing Service to form a nucleus for the computational sub-division of the new organisation. It is anticipated that the computational needs of the Admiralty will be met by outside organisations and the mathematical needs by an even larger use of the service of consultants, working under the general direction of the Director of Physical Research, Admiralty.

Among the consultants employed during the war were Dr. N. ARONSZAJN, Professor W. G. BICKLEY, Dr. L. J. COMRIE (Scientific Computing Service, Ltd.), Professor E. T. COPSON, Dr. J. COSSAR, Dr. A. ERDÉLYI, Professor P. P. EWALD, Dr. H. KOBER, Dr. J. MARSHALL, Dr. J. C. P. MILLER, Professor E. H. NEVILLE.

It is perhaps as well to explain that Admiralty Computing Service started on a scale severely limited by difficulties of staff recruitment; its main object in its early stages was to obtain the numerical results required in problems of war research and to make those results available to the particular establishment concerned as early as possible. Publication of reports was then a secondary matter, and, in fact, it has generally been so regarded as far as the purely computational work is concerned. For this reason, it was not until the beginning of 1944 that the numbering of Admiralty Computing Service reports was systematised in the SRE/ACS series. All reports issued prior to that date, either with an NAO (=Nautical Almanac Office) serial number or with an SRE/MA (=Headquarters) reference number, were renumbered in the new series and will be referred to here by those numbers.

All the reports mentioned below were issued by the Admiralty Computing Service of the Department of Scientific Research and Experiment (Admiralty), Great Britain; generally the computational reports were prepared and reproduced by H. M. Nautical Almanac Office, while the mathematical reports were edited and reproduced at Headquarters.

The following 21 SRE/ACS Reports have already been reviewed in *MTAC* under the heading of Recent Mathematical Tables (one under UMT):

ACS	RMT	No.	p.	ACS	RMT	No.	p.
19	262	13	36	65	226	12	446
21	260	13	35	68	206	11	424
22	267	13	39	71	206	11	424
31	260	13	35	82	252	13	31
39	260	13	35	90	293	14	80
46	260	13	35	91	[UMT 41]	13	52
47	334	16	175	93	277	14	70
52	268	13	40	97	333	16	174
53	206	11	424	102	288	14	76
55	260	13	35	108	352	17	215
62	266	13	38				

There follows a summary of 19 other items of work (ACS **7**, **8**, **9**, **18**, **20**, **26**, **37**, **40**, [**47**], **51**, **80**, **89**, **96**, [**97**], **101**, **106**, **107**, **109**, **110**, **111**, **112**) which appear to have a certain permanent value to the computer and mathematician. It is hoped to arrange for the publication in full of some of the reports. Suggestions as to those which appear suitable for this treatment will be appreciated. Details of other work, not conveniently described, apart from its background, will appear elsewhere.

Copies of the reports are only available for distribution to Government Departments and similar agencies but arrangements have been made for copies of some of the reports to be deposited with the Editors where they may be consulted. A very limited number of photostat copies of the unpublished tables is available for distribution or loan to institutions or individuals with a special computational requirement.

7. *Summation of Certain Slowly Convergent Series.* Stencilled typescript, on one side of 4 p.; undated but issued January 1943. 20 × 32.5 cm.

This note draws attention to a device which was apparently first applied in computational work by P. P. EWALD in his work on crystal-structure, *Ann. d. Phys.*, v. 64, 1921, p. 253–287, Gesell. d. Wissen., Göttingen, *math.-phys. Kl.*, n. s. v. 3II, no. 4, 1938, p. 55–64. The analytical basis of the device is the JACOBI Imaginary Transformation of Theta-function Theory (see E. T. WHITTAKER & G. N. WATSON, *A Course in Modern Analysis*, fourth ed., 1927, p. 475); there is a physical basis, too, which consists in replacing (taking the electrostatic analogy) point charges by Gaussian space-distributions. Applied to the series $S=\sum_{n=0}^{\infty}(n+\frac{1}{2})^{-\frac{1}{2}}e^{-\frac{1}{10}(n+\frac{1}{2})}$ accuracy comparable with that obtained by summation of 50 terms of the original series may be obtained by taking a single term of one of the two infinite series into which S is transformed, and three terms of the other.

8. *Mechanical Quadratures.* Stencilled typescript, 11 p. + 1 diagram; undated but issued December 1942. 20 × 32.5 cm.

Part I is an exposition of the method of (Approximate or) Mechanical Quadratures in which an estimate for $\int_a^b f(x)w(x)dx$ is given as $\Sigma\lambda_n f(x_n)$ where the x_n are the zeros of the orthogonal polynomials associated with the distribution $w(x)$ in the interval (a, b) and the λ_n are certain constants, called the Christoffel numbers.

Brief discussions are given of the cases when $w(x) = 1$, see RMT92,132; $w(x) = e^{-t}$, $t = x^2$, when the polynomials are the Hermite polynomials, see RMT131,250; $w(x) = e^{-x}$ when the polynomials are the Laguerre polynomials, see RMT252; and $w(x) = x^a e^{-x}$ when the polynomials are the Sonine polynomials.

Part II applies these methods to a particular case of estimating a probability integral of the form $\int_0^\infty f(x)e^{-t}dx$, $t = x^2$.

A third part is in preparation; this will deal with the two-dimensional case and in it an account will be given of some recent Russian work.

The most promising of the methods discussed appears to be the Laguerre case and consequently the definitive table of the λ_n, x_n, referred to above, see RMT252, was prepared.

9. *Table of* $f(x, y) = (2\pi)^{-1}\int_0^{2\pi} e^{-x\cos\theta - y\cos^2\theta}d\theta$. Stencilled typescript, 2 p.; undated but issued April 1943. 20.5 × 33.2 cm.

The function $f(x, y)$ is tabulated for x, y = [0(.25)5; 3–4D], without differences, with a reservation that "the tabular values are unlikely to be in error by more than one unit in the last figure retained."

The table was computed from the relation

$$e^{y}f(x, y) = e^{yj}(x)$$

where j is regarded as an operator such that

$$j^r(x) = \frac{(2r)!}{2^r r!\,x^r}I_r(x)$$

where $I_r(x)$ is the Bessel function of purely imaginary argument. The BAASMTC values of $i_r(x) = x^{-r}I_r(x)$ were used in the computation.

18. *Cable Tables.* Stencilled typescript, 10 p.; undated, but issued in August 1943. 33.5 × 40.5 cm. With separate introductory text, mimeographed, 2 p. 20.5 × 33.5 cm.

These tables are a re-issue of earlier tables prepared by H. M. Nautical Almanac Office prior to the formation of Admiralty Computing Service.

The problem is connected with the form of a heavy cable in a uniform stream, but it is thought that the tables are of more general interest and may have other applications.

Defining

$$\ln f(\theta) = \int_0^\infty \frac{\sin U\,dU}{\cos U + \mu \sin^2 U},$$

the quantities tabulated are:

$$y/\bar{y} = \mu\{f(\theta) - 1\}, \qquad s/\bar{y} = \mu \int_0^\infty \frac{f(U)dU}{\cos U + \mu \sin^2 U},$$

and the difference $(s - y)/\bar{y}$, for the ranges: $\theta = 0(1°)90°$; $\mu = .05, .1(.1).5, .4(.2)2(.5)5(1)12$, with 3D in $y/\bar{y}$ and $s/\bar{y}$, but 4D in $(s - y)/\bar{y}$ for $41° \leq \theta \leq 90°$.

No differences have been provided and interpolation is not always linear, though $(s - y)/\bar{y}$ behaves smoothly even for small μ and θ near 90°, where both $y/\bar{y}$ and $s/\bar{y}$ are not easily interpolable. It is not expected that the last figure will be in error by more than one unit, though no great effort was made to ensure end-figure accuracy.

20. *Trajectories of a Body Moving with Resistance Proportional to the Square of the Velocity.* Stencilled typescript; 6 p. + 1 p. diagrams. 20.2 × 33 cm.

Tabulations are connected with projectile trajectories when the motion is under gravity, with a resistance proportional to the square of the velocity.

The functions tabulated are

$$X = \int_{\theta_0}^{\theta} \frac{\sec^2 \psi d\psi}{F(\psi) - F(\alpha)}, \qquad Y = \int_{\theta_0}^{\theta} \frac{\tan \psi \sec^2 \psi d\psi}{F(\psi) - F(\alpha)}, \qquad S = \int_{\theta_0}^{\theta} \frac{\sec^3 \psi d\psi}{F(\psi) - F(\alpha)},$$

$$T = \int_{\theta_0}^{\theta} \frac{\sec^2 \psi d\psi}{\sqrt{F(\psi) - F(\alpha)}}, \qquad \text{and} \qquad \dot{S} = \frac{dS}{dT} = \frac{\sec \theta}{\sqrt{[F(\theta) - F(\alpha)]}},$$

where $F(\theta) = \sec\theta \tan\theta + \ln(\sec\theta + \tan\theta) = \int_0^\theta \sec^3 \psi d\psi$.

In tables I(a) to II(b), $\theta_0 = 89°$, and since this makes most of the values of X, Y, S, T negative, the quantities actually tabulated are $-X, -Y, -S, -T, \dot{S}$. The tabulations are generally to 3D. No attempt at great accuracy has been made in the calculations. In the main tables it can be stated that

(i) the maximum error possible is ten units in the last figure retained: this error, if it occurs at all, will be systematic and will therefore not enter with its full weight.
(ii) errors of more than three units in the last place are unlikely.

Tables have been prepared for $\alpha = -85°(5°)85°$ with θ as independent variable, and for $\alpha = -90°(5°)-80°$ with T as independent variable.

Table I(a) : $\alpha = -10°(5°) + 85°$, $\theta - \alpha$ from 0 to 10°.
Table I(b) : $\alpha = 70°(5°)85°$, $\theta - \alpha$ small, less than 0°.1.
Table II(a): $\alpha = -85°(5°) + 80°$, θ from $-5°$ (or some larger value depending on α) to 89° for $\alpha \leq -50°$ and to 85° for $\alpha \geq -45°$.
Table II(b): α from $-85°$ to 85°, θ from 85° to 90°.
Table III : $\alpha = -90°(5°) - 80°$, argument $T = 0(.1)1(.2)2.8$.
Table IV : an auxiliary table giving $F(\theta)$ for $\theta = [0(0°.1)70°;\ 4D]$, and $\cos^2\theta F(\theta)$ for $\theta = [60°(0°.1)90°;\ 4D]$.

The report contains instructions for the use of the tables and indicates the method of computation.

26, 106. E. T. Copson, ***The Asymptotic Expansion of a Function defined by a Definite Integral or Contour Integral.*** Mimeographed, 1943, 45 p. **106.** Second revised ed., 1946, 63 p. 20.2 × 33 cm.

This monograph gives an account of the methods used in the asymptotic evaluation of integrals. It includes the method of integration by parts, Laplace's method, Kelvin's prin-

ciple of stationary phase, the use of Watson's lemma, method of steepest descent and saddle point method. Applications are made to the gamma function, incomplete gamma functions, Bessel functions, scattering of sound waves, Airy integrals, Legendre and Hermite polynomials, and other functions.

In the second edition the section on Airy's integral has been revised in order to bring the notation in line with that used by the British Association Tables, and an application to a problem in probability has been added. The Bibliography (with brief notes on the scope of each item) will be of assistance to those who require further information.

37. J. M. Jackson, *An Electronic Differential Analyser*. Mimeographed, 1944. 19 foolscap leaves + 4 plates. Reprinted by the Navy Department, Washington, D. C., Office of Research and Inventions, July, 1946.

The processes required in a differential analyser, e.g. addition (subtraction), multiplication, integration, and differentiation, can all be performed by simple electronic circuits. The chief drawback is that the lower frequencies being differentiated suffer attenuation and phase shift, and that the process of integration can occur only for a limited time. Both these troubles can be appreciably relieved by liberal use of negative feed back.

A differential analyser was built on these lines and, within the limits of its accuracy, about 5%, gave good service. The main advantages, apart from the low initial cost, are the rapidity of operation, and the simplicity of setting up, each unit, whether adder, multiplier, or integrator, being simply plugged into the correct position in the chain. The output of the machine operated a pen and ink recorder. It is felt that a more detailed investigation of the possibilities of the method should be made in cooperation with experienced electronic engineers.

40. *Rangefinder Performance Computer*. Stencilled typescript, July 1944, 3 p. 20.4 × 33 cm.

This device calculates the mean error and root mean square error made by an operator being trained in the use of a mechanism such as a range-finder. It thus reduces greatly the labor both of selecting trainees according to natural ability and of assessing the value of their training.

The true reading, and that obtained from the operator's use of his mechanism, are fed simultaneously into a differential gear which rotates a uniselector. The instantaneous error appears in the uniselector expressed as an integer up to ± 24. At instants determined either by a timing device or by the operator himself, this error is transferred to the computing mechanism, which, by uniselectors and relays, adds it to the sum of all previous errors, and by means of a built-in table of squares, adds its square to the sum of the squares of the previous errors. It also counts the number of errors inserted. These operations require up to 4 seconds; the results, including the sign, are shown on an illuminated display panel. The machine can accommodate up to 100 errors, or can be set to stop automatically at any previous number. It can be reset to zero almost instantaneously.

The calculation is completed either by a hand machine or by inserting the numbers from the display panel into a suitable network of inductance potentiometers.

[47]. *Tabulation of the Function* $f(x,y) = \int_0^\infty e^{-k}[J_0(kx)\cosh(ky) - 1]dk/\sinh k$. Photo-offset print of handwriting and machine-printed tables, February 1945. 9 p. + 1 folding diagram. 21 × 34.5 cm. See RMT **334**.

The function $f(x, y)$ is the solution to a two-dimensional potential problem, being that solution in the strip $0 \leq y \leq 1$ of the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} = 0,$$

which satisfies the conditions $\frac{\partial f}{\partial y} = 0$ on $y = 0$ for $x \neq 0$, $\frac{\partial f}{\partial y} = (1 + x^2)^{-3/2}$ on $y = 1$, and $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sim O\left(\frac{1}{x^2 + y^2}\right)$, both near the origin and for large values of x.

The function is tabulated for $x = 0(.1)5$; $y = [0(.1)1; 4D]$, with second differences in both the x- and y-directions. There is also a diagram showing the contours in the x, y-plane for $f(x, y) = -1(.1) + .3$, and the values of $f(x, y)$ for $x = 0(.2)5$; $y = 0(.2)1$ obtained by the use of relaxation technique.

Expansion as a series gives:

$$f(x, y) = \sum_{n=1}^{\infty} [\{x^2 + (2n - y)^2\}^{-\frac{1}{2}} + \{x^2 + (2n + y)^2\}^{-\frac{1}{2}} - 1/n]$$

and this was used, in various forms, to obtain values to 6D on the lines $x = 0$, $x = 5$, $y = 0$ and $y = 1$ and for one or two interior points as a check. Computation for the remaining interior points at interval .2 in both x and y was performed by applying the technique of relaxation to solve the simultaneous equations arising from the finite-difference equivalent of the partial differential equation defining the function. As far as it is known this is the first application of the relaxation method specifically including in the equations to be solved the corrections for the effect of higher order differences. Their inclusion allows the use of a larger interval than would otherwise be permissible. The excellence of the agreement of the independent calculations and the method of computation suggests that no value is in error by much more than one unit in the last figure retained.

51. *Table of Angular Quarter Squares.* Photostat, July 1944. 11 p. 20 × 25 cm.

This table gives, for the range 0(3′)100°, the angular equivalent in degrees and minutes of the radian measure of the quarter square of the radian measure of the argument. It was prepared for the Admiralty Compass Observatory for use in analysing compass deviations.

The preparation of the table is trivial and it was actually built up from a constant second difference on a sexagesimal National accounting machine, the machine being arranged to print the final copy directly.

80. *Probability Charts for Destructive Tests.* Sheets mimeographed on one side only, November 1945. 5 p. of introductory text, 4 folding charts. 20.2 × 33 cm.

Tables are given to 2D of $\log \left[1000\, C^{N-M}_{n-c}\, C^{M}_{c} \Big/ C^{N}_{n}\right]$ as a function of n for the range of values $N = 50, 100, 200, 400$; $M = 0(1)10, 12, 15$; $c = 0(1)2$; $n < \frac{1}{4}N$.

Values for the limiting case $N = \infty$ are also tabulated as a function of n/N.

89. *Solution of Integral Equations occurring in an Aerodynamical Problem.* Photo-offset print of handwriting and machine printed tables, July 1945. 17 p. 20.5 × 33 cm.

The actual tables consist of 2 p. only, on which are tabulated:

in Table I, $K'(x)$, $G(x)$, $G'(x)$, $S(x)$, $U(x)$ and $W(x)$ for $x = [0(.2)10; 5D]$;

in Table II, $\omega'(x)$, $\lambda'(x)$ and $f(x)$ for $x = [0(.2)10; 5D]$. These functions are defined as follows:

$$K'(x) = \frac{2\sqrt{2}}{\pi} \frac{1}{2x\sqrt{x+2}} \{(1 + x)E_1(k) - F_1(k)\}$$

$$G(x) = \frac{2\sqrt{2}}{\pi} \frac{1}{\sqrt{x+2}} F_1(k), \qquad G'(x) = \frac{2\sqrt{2}}{\pi} \frac{1}{2x\sqrt{x+2}} \{E_1(k) - F_1(k)\},$$

where $E_1(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi$, and $F_1(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi$ are the elliptic integrals of the second and first kind respectively and $k^2 = x/(2 + x)$.

From these S, U and W are derived by means of the equations

$$S(x) = K'(x) - \int_0^x K'(x - y)S(y)dy, \qquad U(x) = G(x) - \int_0^x G(x - y)S(y)dy,$$

$$W(x) = - U'(x) = \{G(0)S(x) - G'(x)\} + \int_0^x G'(x - y)S(y)dy.$$

For the functions in Table II,

$$\omega'(x) = \frac{2\sqrt{2}}{\pi} \frac{1}{\sqrt{x + 2}} \{(x + 2)E_1(k) - F_1(k)\},$$

$$\lambda'(x) = \frac{\sqrt{2}}{\pi} \frac{1}{x\sqrt{x + 2}} \{(2x^2 + 4x + 1)E_1(k) - (2x + 1)F_1(k)\},$$

and $f(x)$ is defined by the integral equation: $f(x) = \omega'(x) - \int_0^x \lambda'(x - y)f(y)dy$.

The problems from which the two tables arise are essentially the following:

(i) To solve the integral equation $g(z) = \int_0^z \frac{(z + 1 - t)}{\{(z - t)(z + 2 - t)\}^{\frac{1}{2}}} h(t)dt$ for $h(z)$ in terms of the general function $g(z)$. The solution is

$$h(z) = w(z) - \int_0^z S(z - t)w(t)dt$$

where $w(s) = \frac{\sqrt{2}}{\pi} v'(s)$, and $v(s) = \int_0^s g(z)(s - z)^{-\frac{1}{2}}dz$ and so can be obtained by quadrature for any given function $g(z)$.

(ii) To determine the function $f(z)$ from the equations $f(z) = \int_0^z k_2(z - t)h(t)dt$, where $1 = \int_0^z k_1(z - t)h(t)dt$, and $k_1(x) = \frac{(x + 1)^2}{\{x(x + 2)\}^{\frac{1}{2}}}$, $k_2(x) = \frac{(x + 1)}{\{x(x + 2)\}^{\frac{1}{2}}}$.

Simple elimination leads to the equation

$$\int_0^z k_1(z - t)f(t)dt = \int_0^z k_2(t)dt = \{z(z + 2)\}^{\frac{1}{2}},$$

a form identical with the first problem, the general solution of which could be used to compute $f(z)$.

The main interest in the report lies in the method of numerical solution for the integral equations of the second kind defining $S(z)$ and $f(z)$. A simple direct method is developed by replacing the integral by an accurate quadrature. It is also shown that the following explicit formulae for $S(z)$, $U(z)$ and $f(z)$ may be obtained by the application of the Laplace transformation:

$$S(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{u(1-z)}}{K_1^2(u) + \pi^2 I_1^2(u)} \frac{du}{\sqrt{u}}, \qquad U(z) = \int_0^\infty \frac{e^{-uz}}{K_1^2(u) + \pi^2 I_1^2(u)} \frac{du}{u^2}$$

and $f(z) = e^{-.6435z}(1.2120 \cos .5012z + .1898 \sin .5012z) - \int_0^\infty \frac{e^{-uz}}{K_1'^2(u) + \pi^2 I_1'^2(u)} \frac{du}{u^2}$. Here K_1, I_1 are the Bessel functions of pure imaginary argument, primes denote derivation with respect to the argument, u, and the first term in $f(z)$ arises from the fact that $K_1'(z)$ has 2 (and only 2) zeros in the complex plane: $z = - .6435 \pm .5012i$. These expressions were used to provide an independent check on the accuracy of the numerical solution, and it is difficult to conceive a check which is more satisfactory. They do not, however, appear to be particularly suitable for systematic numerical computation.

This problem has had a certain amount of publicity owing to the disparity between the original estimate of 300 man-years, made by the Establishment concerned, and the actual time of 50 hours taken by Admiralty Computing Service.

96. *Computation of an Integral occurring in the Theory of Water Waves.* Photo-offset of handwriting and machine printed tables, September 1945. 6 p. 20.5 × 32 cm.

The theory demands numerical values of

$$C(u) = \int_0^\infty \cos \sigma u \operatorname{sech} kh\, d\sigma$$

where $\sigma^2 = gk \tanh kh$, for various values of the parameter h, g being a constant. It is readily seen that a single table of

$$f(v) = \sqrt{\frac{h}{g}}\, C(v) = \int_0^\infty \cos vy \operatorname{sech} x\, dy$$

will enable $C(u)$ to be calculated quickly and easily for any combination of h, g and u.

$f(v)$ has accordingly been tabulated for $v = [0(.01)5(.1)10;\ 4D]$; no differences are given, but first differences are always small.

A short auxiliary table is included of sech x with argument y where $y^2 = x \tanh x$; sech x to 4D, with δ^2 for $y = 0(.1)3.5$.

[97]. *Tables of the Incomplete Airy Integral.* Photo-offset typescript and machine printed figures, April 1946. 5 p. of tables (photo reduced machine figures) and 10 p. of introductory text. 20.8 × 32.6 cm. See RMT **333**.

This report contains tables of the integral

$$F(x, y) = \frac{1}{\pi}\int_0^\pi \cos (xt - yt^3)dt$$

for $x = -2.5(.1) + 4.5$; $y = [0(.02)1;\ 4D]$. No Δ given, but the table can be interpolated in the x-direction using second differences; accurate interpolation in the y-direction requires the use of fourth differences, but the use of second differences only will give rise to a maximum error of 2 units only. The last figure tabulated should not be in error by more than one unit, and all the computations have been done in units of the seventh decimal, though two of these were often lost in the course of the integrations.

The title of the report is obtained by considering the function in the form:

$$F(x, y) = \frac{1}{\pi Y}\int_0^{\pi Y} \cos (XT - \tfrac{1}{3}T^3)dT$$

where $Y = (3y)^{1/3}$ and $X = x/Y$. It is thus seen that, for the purpose of extending the tabulation of the complete integral, the present choice of variables is not ideal.

The report contains an account of the methods of computation used and gives the necessary ascending and asymptotic series. The principal method of solution was the direct numerical integration in the x-direction of the differential equation satisfied by the function

$$\frac{\partial^2 F}{\partial x^2} + \frac{x}{3y} F = \frac{1}{3\pi y} \sin (x\pi - y\pi^3).$$

A detailed account is given of the application of the National accounting machine to the integration of second-order differential equations of this type; the method is one which will have great value for use with automatic digital computing machines.

Special methods had to be developed for use for small values of y, and generally the function is one of surprising computational difficulty.

101, 107, [109, 110, 111]. H. Kober, *Dictionary of Conformal Representations, Parts I–II.* Mimeographed on one side of leaves, 1945, 1946. 36 and 48 leaves. 20.2 × 33 cm.

Each page of this dictionary is divided into two columns; that on the left gives diagrams or formulae relating to z-plane while that on the right gives the corresponding diagrams or formulae for the $w = f(z)$ plane.

In Part I the cases of the linear ($f(z) = az + b$) and bilinear ($f(z) = (az + b)/(cz + d)$) transformations are discussed. For instance, explicit formulae giving the actual transformation which carries any two non-intersecting circles into two concentric circles are given.

In Part II, Algebraic functions and z^{α} for real values of α are discussed. Included in this part is a section on JOUKOVSKI's transformation $w = az + b/z$ and its generalisations.

This dictionary will be completed by the issue of three further parts:

109 Part III, *Exponential Functions and some related Functions*,
110 Part IV, *Schwarz-Christoffel Transformations*,
111 Part V, *Higher Transcendental Functions*.

112. ALAN BAXTER (1910–1947), *The Fourier Transformer*. 1947.

This is a machine of mixed electrical and mechanical design which evaluates the Fourier transforms $C(n) = \int_0^\infty f(n) \cos nx\, dx$, and $S(n) = \int_0^\infty f(n) \sin nx\, dx$ of a given function $f(x)$. The integrations are carried out electrically but the selection of the wave-number n is mechanical. The input function is followed manually and the motion translated into a proportional A.C. voltage (50 c/s) by an inductance potentiometer. A power supply of equal voltage is derived from a servo-operated Variac transformer. This feeds simultaneously 22 magslip resolvers, each of which multiplies the voltage by the sine and cosine of the particular angle at which its rotor lies. These voltages derived from the magslip resolvers, proportional to $f(x) \cos nx$ and $f(x) \sin nx$, are integrated by modified sub-standard K.W.H. meters.

Each magslip rotates continuously during the transit of the input function, to produce the appropriate angle nx, the angles being selected by a system of gear boxes. The range of wave numbers n is from $\frac{1}{16}$ to 128, a figure which can be further increased if the function is subdivided. One traverse of the function occupies 10 minutes, and produces simultaneously 22 cosine integrals and 21 sine integrals whose wave numbers cover a range of 4:1. The full range from $\frac{1}{16}$ to 128 is covered, if required, by repeated following of the input function.

Additional dispositions of the gear boxes enable the density of wave-numbers in any particular range to be increased at first fourfold, and then a further tenfold, allowing for close investigation of any particularly interesting regions of the transform. It is hoped that the errors in the transform will be less than 1 per cent of its peak value.

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[1] Other aspects of the work of Admiralty Computing Service are described in an article by the present authors, in *Nature*, v. 157, May 4, 1946, p. 571–573; see *MTAC*, v. 2, p. 188. See also A. ERDÉLYI and JOHN TODD, *Nature*, v. 158, 1946, p. 690; ACS **115**.

RECENT MATHEMATICAL TABLES

Seven reviews of RMT are to be found in our introductory article "Admiralty Computing Service," **7, 9, 18, 20, 80, 89, 96.**

400[C].—JOSEF KŘOVÁK, *Achtstellige Logarithmische Tafel der Zahlen. Osmimístné Logaritmické Tabulky Čísel*. Prague, Geographic Institute of the Minister of the Interior, 1940. iv, 26 p. 14.6 × 21.1 cm.

This little pamphlet is divided into two parts. In the first part (p. 2–14) the arguments are the logarithmic mantissae from 0000 to 6389 corresponding to the numbers 10 000 000 to 43 541 160. In the second part (p. 15–26) are given the mantissae of log N, for $N = 4340(1)10009$. In columns headed d, of each part, are the greatest and least differences which arise in successive lines. Examples in German and Czechish illustrate the interpolation process for getting the 8-figure logarithm of any number.

401[D].—JOSEF KŘOVÁK, *Sechsstellige Tafeln der natürlichen Werte der Funktionen Sinus und Cosinus für Winkel in Zentesimalteilung. Šestimístné Tabulky přirozených hodnot Funkcí Sinů a Cosinů úhlů v setinném dělení.* Prague, Geographic Institute of the Minister of the Interior, 1943?, xii, 52 p. + two slips (in German) listing errors in the first two editions as well as in the current volume. 14.9 × 20.8 cm.

From the errata slips of this undated volume it would seem as if the edition before us was at least the third. And from the preface of the first edition (1941) of Křovák's *Natürliche Zahlen der Funktion Cotangens*, of which the second edition (1943) was reviewed in RMT **362**, it is clear that the first edition of the six-place table appeared in 1940.

The table gives the natural values of sine and cosine for each centesimal minute, with differences, each page being devoted to 1^g.

On p. 51, there are "Tafeln für die Berechnung zehnstelliger Logarithmen der Zahlen N und umgekehrt," explanations following on p. 52.

R. C. A.

402[D].—JOSEF KŘOVÁK, **a.** *Zwölfstellige Tafeln der trigonometrischen Funktionen. Dvanáctimístné Tabulky trigonometrických Funkcí.* Prague, Landesvermessungsamt Böhmen und Mähren, second ed., 1944. viii, 271 p. + 2 loose sheets of tables. 21.8 × 29.6 cm.

b. *Koeffizienten zur Berechnung der zweiten Interpolationsglieder. Koeficienty pro výpočet druhých interpolačních členů.* Second ed., Prague, 1944. vi, 26 p. 21.8 × 29.6 cm.

a. This table, which appeared originally in 1928, is a twelve-place sexagesimal table of sines and cosines for each 10″ of the quadrant, and of tangents, at a similar interval, 0 to 45°. It is the first twelve-place table of the kind. But among recent tables Andoyer's fifteen-place tables of these three functions (1915–1916), throughout the quadrant, at interval 10″, will be recalled.

Each page is devoted to 10′ and columns with first and second differences (d_1, d_2) are given.

The companion five-place table **b**, which appeared also with the original edition, facilitates interpolations. There are also two loose-sheet tables for calculating second and third differences: $-z(1-z)d_2/2!$, $z(1-z)(2-z)d_3/3!$.

The author tells us that his values of the sine and cosine functions, insofar as they were correct, were rounded off from the fifteen-place Rheticus-Pitiscus table of 1613, *Thesaurus Mathematicus sive canon Sinuum ad Radium 1.00000.00000.00000 et ad dena quaeque Scrupula secunda Quadrantis. . . .* The values of the tangents (cotangents) were derived, with the aid of calculating machines, from the quotients of the values for sines and cosines. All values for each function were tested by differences, and by comparison with the tables of Andoyer.

R. C. A.

403[D].—J. T. PETERS, *Sechsstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Neugrades.* Ninth ed., Berlin, Wichmann, 1944. iv, 512 p. 18.9 × 25.3 cm.

The preface of this work is dated "Berlin, im August 1938" and this is the year when the first edition was published (iv, 512 p.). A third edition with corrections appeared in 1940 and a seventh edition in 1943. The later "editions" are presumably nothing but "reprints." They reflect however the great popularity of the centesimal division of the quadrant in German war applications.

Pages 2–501 contain the numerical values of the trigonometric functions sin, tan, cos, to 6D, cot to 6S, at interval $0^g.001$ or 10 centesimal seconds. On each page are the values of the functions for 10 centesimal minutes. Differences and proportional parts are given at the foot of each page. On pages 2–21 we also find here a 6S auxiliary table w cot w for each $0^g.01$ to $2^g.00$.

Apart from some worked out examples and constants, the final pages (503–512) contain a number of tables for changes between the sexagesimal and centesimal units, between centesimal units and time, and mils, and for the improvement of the Gauss-Krüger projection.

Typographically the pages are very unattractive.

The Czechoslovakian volumes of the six-place centesimal tables of cotangents, and of sines and cosines, have been already referred to in RMT **362** and **401**.

R. C. A.

404[D].—J. T. Peters, *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Neugrades.* Berlin, Verlag des Reichsamts für Landesaufnahme, 1941. viii, 544 p. 17 × 26 cm.

This was probably the last printed volume of tables coming from Peters' hands. The preface is dated March 1941; he died in the following August. The fine appearance of this 7-place table happily contrasts with that of the 6-place table of the previous review. As in that table the functions are the sin, tan, cot, cos and in the first 2^g $w \cot w$ is also tabulated. Throughout the quadrant the interval is $0^g.001$. Proportional parts are given at the bottom of each page. From 2^g on, the first two decimals of each value are printed at the head of each column. The main table ends on p. 521.

Pages 525–544 are devoted to trigonometric formulae, series, solutions of equations, least squares, mathematical and geodetic constants, tables indicating the relations between sexagesimal and centesimal degrees, centesimal degrees and radians.

Thus endeth the twentieth major mathematical table prepared under the direction of Peters. See *MTAC*, v. 1, p. 168f. The 7-place table of the trigonometric functions for each 0°.001 in the quadrant were sent forth by Peters in 1918, and reprinted in 1930 and 1938. For the American edition, 1942, see *MTAC*, v. 1, p. 12f.

R. C. A.

405[D, E, V].—ÎA. M. Serebriĭskiĭ, "Obtekanie Krylovykh Profileĭ Proizvol'noĭ Formy" (Flow past an aerofoil of arbitrary form), Akad. N., SSSR., Moscow-Leningrad, *Inzhenernyĭ Sbornik* (*Engineering Review*), v. 3, no. 1, 1946, p. 105–136. 16.4 × 25.7 cm. See RMT **194**, v. 1, p. 390.

There are the following tables:

T. 1, p. 111, $\cosh \psi$, defined by equations: $x = \cosh \psi \cos \theta$, $y = \sinh \psi \sin \theta$, for $x = 0(.2).6(.1).8(.05).9$, $.93(.02).97(.01).99(.005)1(.01)1.05$; $y = 0(.01).02(.02).22$. Tables 1–9 are to 3D. $\operatorname{Cosh} \psi = [w \pm (w^2 - x^2)^{\frac{1}{2}}]^{\frac{1}{2}}$, where $w = \frac{1}{2}(1 + x^2 + y^2)$.

T. 2, p. 114, $\psi_n(\theta) = [\frac{1}{2}(1 + \cos \theta)]^n$, for $\theta = 0(5°)20°(10°)180°$; $n = 1(1)10(5)20(10)40(20)80$. Same ranges of θ, n in T. 4–6.

T. 3, p. 117, $\psi_n(\theta) \sin \theta$, for $\theta = 0(5°)20°(10°)180°$; $n = 1(1)10(5)20$. Same ranges of θ, n in T. 7–9.

For **T. 4** and a few others we need the definition of a *conjugate*; as follows: If a function F is expressed in the form $F = \sum_{n=0}^{\infty} A_n \cos n\theta/\xi^n$, then its *conjugate* is $E[F] = \sum_{n=0}^{\infty} A_n \sin n\theta/\xi^n$. [E.g., taking the above $\psi_n(\theta) = [\frac{1}{2}(1 + \cos \theta)]^n$, we have $\psi_2(\theta) = [\frac{1}{2}(1 + \cos \theta)]^2 = \frac{1}{4}(1 + 2 \cos \theta + \cos^2 \theta) = \frac{1}{4}(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta) = \frac{1}{4}(\frac{3}{2} \cos 0° + 2 \cos \theta + \frac{1}{2} \cos 2\theta)$; $E\psi_2(\theta) = \frac{1}{4}(\frac{3}{2} \sin 0° + 2 \sin \theta + \frac{1}{2} \sin 2\theta) = \frac{1}{2} \sin \theta + \frac{1}{8} \sin 2\theta$.]

T. 4, p. 122, conjugate of $\psi_n(\theta)$.

T. 5, p. 123, derivative of $\psi_n(\theta)$.

T. 6, p. 124, derivative of conjugate of $\psi_n(\theta)$.

T. 7, p. 126, conjugate of $\psi_n(\theta) \sin \theta$.

T. 8, p. 127, derivative of $\psi_n(\theta) \sin \theta$.

T. 9, p. 128, derivative of conjugate of $\psi_n(\theta) \sin \theta$.

There are graphs of the various functions tabulated T. 1–9.

T. 10, p. 129, values of coefficients $A_{k,n}$ in the development $\psi_n(\theta) = \sum_{k=0}^{n} A_{k,n} \cos k\theta$, with the recurrence formula $A_{k,n} = \{[n - (k - 1)]/(n + k)\}A_{k-1,n}$ for $k = 0(1)24$; $n = [1(1)10(5)20(10)40(20)80; 4D]$.

On p. 130, similar values of coefficients $B_{k,n}$ in the development $\psi_n(\theta) \sin \theta = \sum_{k=1}^{n+1} B_{k,n} \sin k\theta$, where $B_{k,n} = .5(A_{k-1,n} - A_{k+1,n})$, $B_{0,n} = 0$, $B_{1,n} = A_{0,n} - .5A_{2,n}$, for $k = 1(1)10(5)20$, $n = [1, 2; 4D]$.

S. A. J.

406[E].—HARVARD COMPUTATION LABORATORY, *Fifteen-place table of e^{-x}.* Bureau of Ships Computation Project. *Publication* no. 20, July 1945. iv, 6 leaves. 21.5 × 27.8. Out of print.

This is a table of e^{-x}, for $x = [0(.05)30; 15D]$. C. E. VAN ORSTRAND, Nat. Acad. Sci., *Memoirs*, v. 14, no. 5, 1921, gave in T. VI, the value of e^{-x}, $x = [0(.1)50; 33–48D]$.

407[E].—H. S. UHLER, "Special values of $e^{k\pi}$, cosh $(k\pi)$ and sinh $(k\pi)$ to 136 figures," Nat. Acad. Sci., *Proc.*, v. 33, Feb. 1947, p. 34–41. 17.3 × 25.7 cm.

T. I. $e^{\pi/m}$, $\pm m = 1(1)4(2)8(4)16(8)32(16)64, 96, 192$. T. II–III, Sinh (π/m), Cosh (π/m), for $+m$. Various checks for accuracy are stated.

408[E, O].—K. G. HAGSTROEM, "Un problème du calcul stochastique," *Försäkrings Matematiska Studier tillägnade Filip Lundberg*, Stockholm, 1946, p. 104–127, table p. 127. 18.5 × 24.4 cm.

This volume of studies in actuarial mathematics is dedicated to Dr. Lundberg, former general manager of the Life Insurance Co., De Färenade, and Sickness Insurance Co. EIK, on his 70th birthday Dec. 31, 1946. There is a "table of $u(z)$," arranged however as a 3D table of z for $u(z) = 0(.01)2.4$; $z = u^{-1} \ln \cosh u = \frac{1}{2}u - \frac{1}{12}u^3 + \frac{1}{45}u^5 - \frac{17}{2520}u^7 + \frac{31}{14175}u^9 - \cdots$, $u(z) = 2z + 1.3333z^3 + 1.2444z^5 + 1.2529z^7 + 1.2935z^9 + \cdots$.

409[F].—ALBERT GLODEN, "Factorización de numeros de la forma $x^4 + 1$," *Euclides*, Madrid, v. 5, 1945, p. 620–621. 16.6 × 24.1 cm.

In this note the author republishes some of his factorizations of $x^4 + 1$ (*MTAC*, v. 2, p. 211). Unlike the previous table, this table lists only those factorizations which have been completely decomposed into primes. The variable x ranges irregularly over 132 values between 1008 and 1500.

D. H. L.

410[F].—ALBERT GLODEN, *Table des solutions de la congruence $x^4 + 1 \equiv 0$* (mod$p$) pour $350.000 < p < 500.000$. Luxembourg, author, 11 rue Jean Jaurès, and Paris, Centre de Documentation Universitaire, 1946, 42 p. 21.8 × 27.4 cm. Offset print.

This table is an extension of four previous tables of CUNNINGHAM ($1 < p < 100000$), HOPPENOT ($100000 < p < 200000$), GLODEN ($200000 < p < 300000$) and DELFELD ($300000 < p < 350000$), see *MTAC*, v. 2, p. 71–72, 210–211. As in the previous tables, this table gives only the two solutions x_1, x_2 of

$$x^4 \equiv -1 \pmod{p}, \qquad p = 8k + 1,$$

which are less than $\frac{1}{2}p$, the other two being the negatives of these. The method of construction of the table is as follows. First, one compiles two lists of numbers of the forms $a^2 + b^2$ and $c^2 + 2d^2$. In each list each prime p of the form $8k + 1$ will appear exactly once. In fact

those numbers which appear more than once are composite. The representations

(1) $$p = a^2 + b^2 = c^2 + 2d^2$$

lead at once to the four solutions

$$x \equiv \pm d(a \pm b)/ac \qquad (\text{mod } p).$$

To reduce these values to integers modulo p involves the solution of a single linear congruence for t

$$act \equiv 1 \ (\text{mod } p).$$

This procedure appears to be somewhat shorter than the one earlier described by the author (*MTAC*, v. 2, p. 71–72).

Under the heading of applications the author gives complete factorizations of nine numbers of the form $x^4 + 1$ (RMT **366**). It may be of interest to point out that another application of this table is its use to rediscover the quadratic partitions (1). In fact we may use the tabulated values of x_1, x_2 to construct l and m by:

$$m \equiv \tfrac{1}{2}(x_2 - x_1) \qquad (\text{mod } p),$$
$$l \equiv m(x_2 + x_1) \qquad (\text{mod } p).$$

If we now form the sequence $r_1, r_2, \cdots$ by Euclid's algorithm

$$p = q_0l + r_1, l = q_1r_1 + r_2, \cdots,$$

the first two r's less than $\sqrt{p}$ are values of a, b such that $p = a^2 + b^2$. The numbers c, d may be found in like manner from p and m. It is worth noting that the table gives a list of primes of the form $8k + 1$ between 350000 and 500000.

D. H. L.

411[F].—Irving Kaplansky & John Riordan, "The problème des ménages," *Scripta Mathematica*, v. 12, 1946, publ. Jan. 1947, p. 113–124. 16.7 × 24.8 cm.

This interesting paper on a problem due to Lucas,[1] concludes with a small table of "ménages numbers" $u_{n,x}$ for $x = 0(1)n$ and for $n = 2(1)10$. These numbers may be defined as follows. Let

(1) $$a_1, a_2, a_3, \cdots, a_n$$

be a permutation of $1, 2, \cdots, n$. The integer a_k is said to be in a forbidden place if it is in either the k-th or $k + 1$st place in (1). If (1) has precisely x elements in forbidden places it may be said to be of type x. The ménages number $u_{n,x}$ is the number of permutations of $1, 2, \cdots, n$ of type x. When $x = 0$ this number is the same as the number of ways that n married couples may be seated at a round table, ladies alternating with gentlemen, in such a way that no one sits next to one's spouse. This fact accounts for the name given the function $u_{n,x}$ in general. The famous function $u_{n,0}$ has been tabulated by Lucas[1] for $n = 4(1)20$.

D. H. L.

[1] Edouard Lucas, *Théorie des Nombres*, Paris, 1891, p. 495.

412[F].—Gino Loria, "Sulla scomposizione di un intero nella somma di numeri poligonali," Accad. Naz. Lincei, *Atti, Cl. Sci. Fis. Mat. Nat., Rendiconti*, s. 8, v. 1, 1946, p. 7–15. 18.1 × 26.8 cm.

This note gives three tables showing all partitions of each integer $n \leqq 100$ into not more than (i) four squares, (ii) three triangular numbers and (iii) five pentagonal numbers.[1] The number ν of such partitions is also given. Tables of this kind appear never to have been published before this. The number ν is perhaps of more interest than the actual partitions themselves. It is surprising to find that the tables are quite unreliable. A complete recalculation shows the following errata.

Table I

N	
28	*for* $3+3+1$, *read* $3+3+3+1$; *insert* $5+1+1+1$; *for* $\nu = 2$, *read* $\nu = 3$
34	*insert* $4+4+1+1$; *for* $\nu = 3$, *read* $\nu = 4$
52	*insert* $4+4+4+2$; *for* $\nu = 4$, *read* $\nu = 5$
58	*insert* $7+2+2+1$; *for* $\nu = 4$, *read* $\nu = 5$
68	*insert* $5+5+3+3$; *for* $\nu = 3$, *read* $\nu = 4$
71	*delete* $5+4+4+2$; *for* $\nu = 3$, *read* $\nu = 2$
76	*for* $6+2+2$, *read* $6+6+2$
83	*delete* $8+4+1+1$; *for* $\nu = 5$, *read* $\nu = 4$
89	*insert* $6+6+4+1$; *for* $\nu = 4$, *read* $\nu = 5$
93	*delete* $7+$ at end of first line
96	*for* $8+4+2$, *read* $8+4+4$

Table II

16	*insert* $4+2+2$; *for* $\nu = 2$, *read* $\nu = 3$
18	*insert* $3+3+3$; *for* $\nu = 1$, *read* $\nu = 2$
21	*insert* $5+2+2$; *for* $\nu = 3$, *read* $\nu = 4$
22	*insert* $4+3+3$; *for* $5+3+2$, *read* $5+3+1$; *for* $\nu = 2$, *read* $\nu = 3$
23	*for* $4+4+3$, *read* $4+4+2$
27	*insert* $5+3+3$; *for* $\nu = 2$, *read* $\nu = 3$
28	*for* $5+4+3$, *read* $5+4+2$
45	*for* $8+3+3$, *read* $8+3+2$
49	*for* $8+4+3$, *read* $8+4+2$

Table III

24	*for* $2+2+1+1$, *read* $3+2+2+1+1$
28	*for* $3+2+2+1$, *read* $3+2+2+2+1$
53	*for* $3+3+3+2$, *read* $3+3+3+3+2$
62	*for* $5+2+2+2+2$, *read* $5+3+2+2+2$
80	*insert* $4+4+3+3+3$; *for* $\nu = 5$, *read* $\nu = 6$
92	*insert* $6+3+3+3+2$; *for* $\nu = 7$, *read* $\nu = 8$
95	*insert* $6+4+3+2+2$; *for* $\nu = 5$, *read* $\nu = 6$
96	*for* $7+3+3+3+1$, *read* $7+3+3+1+1$
97	*insert* $5+5+4+2$, and $6+4+3+3$; *delete* $5+4+43+2$; *for* $\nu = 6$, *read* $\nu = 7$
100	*for* $4, 4+4+4+3$, *read* $4+4+4+4+3$.

D. H. L.

[1] For positive integers n triangular numbers are given by the formulae $T = \frac{1}{2}n(n+1)$, and pentagonal by $P = \frac{1}{2}n(3n-1)$.

413[I, L, M], [L].—BAASMTC, *Legendre Polynomials*, (*Mathematical Tables, Part-volume A*). Cambridge, Univ. Press, 1946, 42 p. 22 × 28.4 cm. 8s. 6d.
[L, M], J. C. P. MILLER, *The Airy Integral, giving Tables of Solutions of the Differential Equation* $y'' = xy$. (*Mathematical Tables, Part-volume B*). Cambridge, Univ. Press, 1946, 56 p. 22 × 28.4 cm. 10 s. With this Part-v, are also the following Auxiliary Tables on Cards.
[I], *Coefficients in the Modified Everett Interpolation Formula, BAASMTC, Auxiliary Tables*, no. I. Cambridge, Univ. Press, 1946, 1 p. 19.5 × 26 cm. 6d. each, or 5s. per dozen.
[I], *Table for Interpolation with Reduced Derivatives*, BAASMTC, *Auxiliary Tables*, no. II. Cambridge, Univ. Press, 1946, 2 p. 19.5 × 26 cm. 6d. each, or 5s. per dozen.

Some years ago the BAASMTC considered the problem of tables whose size did not justify their publication as bound volumes of the Committee's series, but which it did seem desirable to publish, and decided to issue them as 'Part-volumes,' in paper covers, with the aim of ultimately combining them into bound volumes of cognate tables. Considerable delay—mainly arising out of the circumstances of the war—has occurred since the inception of the project, but the first two Part-volumes are now issued. Apart from the (not very) stiff paper covers, they are uniform, in page size, typography, arrangement, etc., with the main series.

Part-volume A contains tables of the Legendre polynomials $P_n(x)$ as follows:

$x = 0(.01)1$, $n = 2(1)6$, exact or 7D, with δ^2 or δ_m^2 p. A6, A7.
$x = 1(.01)6$, $n = 2(1)6$, exact, 7 or 8S, with δ^2 or δ_m^2, p. A8–A17.
$x = 0(.01)1$, $n = 7(1)9$, 7D, with δ^2 and δ^4, p. A18, A19.
$x = 1(.01)6$, $n = 7(1)9$, 7 or 8S, with δ^2 or δ_m^2, p. A20–A29.
$x = 0(.01)1$, $n = 10(1)12$, 7D, with δ_m^2 and δ_m^4, p. A30, A31.
$x = 1(.01)6$, $n = 10(1)12$, 6 to 8S, with δ^2 or δ_m^2, and δ^4 (for $x = .5$–1.5), p. A32–A41.
$x = 6(.1)11$, $n = 2(1)6$, 7 or 8S, with δ^2 or δ_m^2, p. A42.

The tables were designed by Dr. L. J. COMRIE, and calculated under his supervision when he was secretary of the Committee. Their nucleus, however, dates back much further, to the Committee's report for 1879, which gave exact values of $P_n(x)$ for $x = 0(.01)1$, up to $n = 7$. $P_8(x)$ for the same range was taken from TALLQVIST[1] and HAYASHI[2] (5 errors in Hayashi), and checked. $P_9(x)$ for this range was calculated by use of the recurrence formula, while the values for $n = 10(1)12$ were calculated from the definitions, independently of any previous work, by Dr. A. J. THOMPSON, who has edited the Part-volume.

For values of x greater than 1 the functions were computed under Comrie's supervision. For the smaller values of n they were built up mechanically from the constant nth difference, while for larger values of n values obtained by use of the recurrence formula were checked by differencing. A short introduction by Dr. Comrie gives further details, and there is a page (A5) of formulae.

The integral

$$W(m) = \int_0^\infty \cos \tfrac{1}{2}\pi(t^3 - mt)dt \tag{1}$$

was introduced by AIRY[3] in 1838, and he gave values for $m = -4(.2)+4$, later extending the table[4] to $|m| = 5.6$. The function $\mathrm{Ai}(x)$ tabulated in Part-volume B is related to the above by

$$\mathrm{Ai}(x) = \tfrac{1}{3}\lambda W(-\lambda x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3 + xt)dt \tag{2}$$

where $\lambda = (12/\pi^2)^{\frac{1}{3}}$.

It can, however, readily be shown that $y = \mathrm{Ai}(x)$ is a solution of the differential equation

$$y'' = xy, \tag{3}$$

and it is more satisfactory to regard the functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ tabulated in the Part-volume as a convenient pair of linearly independent solutions of this differential equation. This equation is an approximation to *any* second order linear differential equation over a limited range not including a singularity. For if the equation be reduced to the canonical form

$$y'' + I(x)y = 0, \tag{4}$$

then near any ordinary point X we have

$$I(x) = I(X) + (x - X)I'(X) + \cdots$$

and inserting this in (4), we recover (3) after a linear change of independent variable. It was this fact that led Dr. HAROLD JEFFREYS to suggest the tabulation. The work has been carried out under the supervision of (and a very large proportion actually done by) Dr.

Miller. Miller contributes also a scholarly, but very readable, introduction, dealing with the history of the function, description of the tables, their computation and checking, methods of interpolation, and the definitions and basic theory of the functions tabulated.

The general solution of (3) will contain two functions, of which $\mathrm{Ai}(x)$ is taken as one, and the other chosen, $\mathrm{Bi}(x)$, is defined by a contour integral. Power series and asymptotic expansions for these functions are given. For large real positive values of x, $\mathrm{Ai}(x)$ tends to zero and $\mathrm{Bi}(x)$ to infinity. For negative real values of x the functions oscillate. It is therefore convenient to write

$$\mathrm{Ai}(x) = F(x)\sin\chi(x), \qquad \mathrm{Bi}(x) = F(x)\cos\chi(x),$$
$$\mathrm{Ai}'(x) = G(x)\sin\psi(x), \qquad \mathrm{Bi}'(x) = G(x)\cos\psi(x).$$

The auxiliary functions F, G, χ, and ψ are slowly varying functions of x, when x is negative and not too small, and are very convenient for tabulation.

The actual tables are:

T. I, $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$, $x = 0(.01)2$, $\mathrm{Ai}(-x)$ and $\mathrm{Ai}'(-x)$ for $x = 0(.01)20$, 8D, with δ^2 or δ^2_m, p. B18–B39;
T. II, log $\mathrm{Ai}(x)$, 8D and $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$, 7D, for $x = 0(.1)25(1)75$, with δ^2 or δ^2_m, p. B40–B42;
T. III, Zeros and turning-values of $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$, 8D, the first fifty of each, p. B43;
T. IV, $\mathrm{Bi}(x)$ and Reduced Derivatives, $x = 0(.1)2.5$, $\mathrm{Bi}(-x)$ and Reduced Derivatives, for $x = 0(.1)10$, 8 to 10D, p. B44–B46;
T. V, Zeros and turning-values of $\mathrm{Bi}(x)$ and $\mathrm{Bi}'(x)$, the first 20 of each, 8D, p. B44;
T. VI, log $\mathrm{Bi}(x)$, 8D and $\mathrm{Bi}'(x)/\mathrm{Bi}(x)$, 7D, for $x = 0(.1)10$, p. B47;
T. VII, Auxiliary Functions, $F(x)$, 7D, and $\chi(x)$, in degrees, 6D, $G(x)$, 7D, and $\psi(x)$, in degrees, 6D, for $x = 0(.1)2.5$, with δ^2_m and γ^4, p. B49;
$F(-x)$ and $G(-x)$, 8D, $\chi(-x)$ and $\psi(-x)$, in degrees, 6D, for $x = 0(.1)30(1)80$, with δ^2, p. B50–B56.

Provision is everywhere made for interpolation to the full accuracy of the tables. Usually this is done by giving second differences (δ^2) or modified second differences (δ^2_m). In some ranges, where the latter are inadequate, we are given γ^4, whose leading term is $\delta^4/1000$, and which contains contributions from higher differences, and formulae whereby the necessary correction can be applied with the aid of coefficients supplied on Auxiliary Table I (see below). With $\mathrm{Bi}(x)$, the tabular interval is too large for interpolation by differences to be convenient. Here we are given reduced derivatives, and Auxiliary Table II facilitates their use.

The introduction contains a short bibliography, and a graph of the functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, their derivatives, and related functions. Formulae are collected on p. B17 and B48.

Dr. Miller is to be thanked for his labours, and congratulated upon their result and the manner of its presentation.

The modified Everett formula is, when $|\gamma_1^4 - \gamma_0^4| > 1$,

$$f(x + \theta h) = \phi f_0 + \theta f_1 + E_0^2\delta^2_{m0} + E_1^2\delta^2_{m1} + M_0^4\gamma_0^4 + M_1^4\gamma_1^4;$$

when $|\gamma_1^4 - \gamma_0^4| \leqslant 1$,

$$f(x + \theta h) = \phi f_0 + \theta f_1 + E_0^2\delta^2_{m0} + E_1^2\delta^2_{m1} + T^4(\gamma_0^4 + \gamma_1^4)$$

with $\phi = 1 - \theta$, and

$$\delta^2_m = \delta^2 - 0.184\delta^4 + 0.038082\delta^6 - 0.00830\delta^8 + 0.0019\delta^{10} - \cdots$$
$$1000\gamma^4 = \delta^4 - 0.27827\delta^6 + 0.06855\delta^8 - 0.01648\delta^{10} + \cdots$$

Auxiliary Table I gives, on one side of a card, for $\theta = 0(.01)1$, E_0^2 and E_1^2 with second differences, 7D, and M_0^4, M_1^4 and T^4, 3D, and ϕ, along with the formulae.

Defining

$$\tau^n = \tau^n f(x) = h^n f^{(n)}(x)/n!,$$

where h is the tabular interval, Taylor's series may be written

$$f(x + \theta h) = f(x) + \theta\tau + \theta^2\tau^2 + \theta^3\tau^3 + \theta^4\tau^4 + \cdots$$

and

$$hf'(x + \theta h) = \tau + 2\theta\tau^2 + 3\theta^2\tau^3 + 4\theta^3\tau^4 + \cdots$$

(on early printings, the *h* of the second formula was unfortunately omitted).

Auxiliary Table II is printed on both sides of the card. One side gives, for $\theta = 0(.01)1$, θ^2, θ^3, θ^4, exact; θ^5, 7D; θ^6, 6D; and θ^7, 5D with one D less for $x \geqslant .5$. The other gives 2θ, $3\theta^2$, $4\theta^3$, exact; $5\theta^4$, 6D; $6\theta^5$, 5D; $7\theta^6$, 4D; and $8\theta^7$, 3D with one D less for $x \geqslant .5$.

The Auxiliary Tables can be purchased separately, and are likely to be found very useful and convenient aids in interpolation. It is to Dr. Miller that the lion's share of credit for their inception and design must go.

W. G. Bickley

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[1] A. H. H. Tallqvist, Finska Vetenskaps-Societeten, *Acta*, v. 32, no. 6, 1904.
[2] K. Hayashi, *Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen*, Berlin, 1930.
[3] G. B. Airy, Camb. Phil. Soc., *Trans.*, v. 6, 1838, p. 379–402.
[4] G. B. Airy, Camb. Phil. Soc., *Trans.*, v. 8, 1849, p. 595–599.

414[K].—Émile F. É. J. Borel & André Chéron, *Théorie Mathématique du Bridge à la Portée de Tous; 134 Tableaux de Probabilités avec leurs Modes d'Emploi. Formules Simples. Applications. Environ 4000 Probabilités. Monographies des Probabilités*, fasc. 5. Paris, Gauthier-Villars, 1940. xx, 392 p. 16 × 25 cm. 175 francs.

The title page description shows the impracticability of attempting to list all the tables in this exhaustive work. These tables fall into 3 main groups: 1, à priori; 2, bidding; 3, play. In the first group the à priori probabilities of all distributions of the 4 suits among the 4 players, of the suits between the 2 partnerships, of the distributions of aces, aces and kings, etc., and the probabilities of voids, singletons, etc., are given. In the second group similar tables are given except now we know the 13 cards in the bidder's hand. The third group of tables mainly covers the probabilities after the dummy has been exposed and we know 26 cards, plus derived tables to cover the cases where part of a suit has been played and we are interested in the distribution of the remainder.

The basic probabilities were calculated exactly as fractions, with the aid of a Pascal triangle complete up to $\binom{52}{26}$ which the authors possess, and are given to varying numbers of decimal places, generally about 6 or 7, but up to 15 places in some cases. Formulae for making the calculations are given in the text, and the use of the Tables in evaluating a hand or selecting the best method of play is illustrated.

The authors point out in the first chapter that ordinary shuffling is quite apt to give results that are far from the random ordering which is assumed in their later calculations.

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415[K].—H. Labrouste & Mme Y. Labrouste, *Analyse des Graphiques résultant de la Superposition de Sinusoïdes.* **a.** *Tables Numériques précedées d'un Exposé de la Méthode d'Analyse par Combinaisons Linéaires d'Ordonnées*. vi, 205 p. 21.9 × 31.6 cm. **b.** *Atlas de Courbes de Sélectivité. Supplément aux Tables Numériques*, 35 plates. 21 × 30.8 cm. Paris, Presses Universitaires de France, 1943. 350 francs.

The problem discussed by the authors is that of harmonic analysis. Suppose that a function $y = y(x)$ is a sum of a finite, though unknown, number of simple oscillations.

How may one find the periods, the phases, and the amplitudes of the terms? The authors' approach to the problems is as follows: Suppose that

$$(1) \qquad y = a \sin(\theta x + \phi) + a' \sin(\theta' x + \phi') + a'' \sin(\theta'' x + \phi'') + \cdots;$$

if such a representation is possible, it is certainly unique. Let us consider $2m + 1$ equidistant points $x_0 - m, x_0 - (m - 1), \cdots, x_0, \cdots, x_0 + m$ symmetric with respect to the point x_0, and let $y_{-m}, y_{-(m-1)}, \cdots, y_0, \cdots, y_m$ be the corresponding ordinates of the curve. We write

$$\theta = 2\pi/n, \qquad \theta' = 2\pi/n', \cdots, \qquad \alpha_m = 2\cos(2\pi m/n), \qquad \alpha'_m = 2\cos(2\pi m/n'), \cdots$$

If we set $Y_\mu = y_\mu + y_{-\mu}$, we get

$$Y_\mu = \alpha_\mu a \sin(\theta x_0 + \phi) + \alpha'_\mu a' \sin(\theta' x_0 + \phi') + \cdots,$$

so that any linear combination

$$R_m = K_0 y_0 + K_1 Y_1 + \cdots + K_m Y_m$$

of the quantities $y_0, Y_1, \cdots, Y_m$ can be written in the form

$$R_m = \rho_m a \sin(\theta x_0 + \phi) + \rho'_m a' \sin(\theta' x_0 + \phi') + \cdots$$

where

$$\rho_m = K_0 + K_1\alpha_1 + \cdots + K_m\alpha_m, \qquad \rho'_m = K_0 + K_1\alpha'_1 + \cdots + K_m\alpha'_m, \cdots.$$

It follows that (with x_0 replaced by x) the phases and the periods of the terms composing R_m are the same as the phases and the periods of the terms of y. If the K's are so chosen that all the numbers ρ except one—say except ρ_m—are very small, the graph of R_m/ρ_m gives the first term on the right of (1). To each period $\theta_0 = 2\pi/n_0$ corresponds a "selective" combination R_m. The numbers K_m are the Fourier coefficients of a function large in the neighborhood of the point θ_0 and small elsewhere. The simplest combinations R_m are $s_m = y_{-m} + \cdots + y_0 + \cdots + y_{m-1} + y_m$. The corresponding multipliers of the amplitudes are then

$$\sigma_m = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_m = \sin(2m + 1)\frac{\pi}{n}\Big/\sin\frac{\pi}{n}.$$

Tables I and II, p. 91–142, of **a** give the values, to 3D, of α_m and σ_m respectively for $m = 0(1)20$, $n = .5(.01)3(.02)5(.05)10(.1)15(.2)25(.5)50(1)100(5)200(10)500$; $m = 21(1)40$, $n = 4(.02)5$[then as above]500; $m = 41(1)50$, $n = 50(1)100(5)200(10)500$. Other tables give values of multipliers corresponding to combinations whose basic element is not $y_\mu + y_{-\mu}$ but $y_\mu - y_{-\mu}$.

In **b** the authors investigate in great detail products (superpositions) of simple combinations and give graphs of the ratio ρ/ρ max as functions of period n. Here ρ is the amplitude multiplier and ρ max is its maximum.

The title-page states that H. Labrouste is a professor in the Faculty of Sciences of the University of Paris and that Mme. Y. Labrouste is an associate physicist at the Institut de Physique du Globe, of the University of Paris.

A. ZYGMUND

University of Pennsylvania

416[K].—L. W. POLLAK, assisted by C. HEILFRON, *Harmonic Analysis and Synthesis Schedules for Three to One Hundred Equidistant Values of Empiric Functions*. (Department of Industry and Commerce, Meteorological Service, *Geophysical Publications* v. 1.) Dublin, Stationery Office, 1947. xxxiii, 118 p. 24.5 × 30.2 cm. £2 s2.

This work is designed to facilitate the work of fitting the Fourier series,

$$y = p_0 + p_1 \cos x + p_2 \cos 2x + \cdots + q_1 \sin x + q_2 \sin 2x + \cdots,$$

to a set of n equidistant values.

The formulae devised for this purpose by F. W. BESSEL more than a century ago require the values of the functions $\cos(2\pi m/n)$, $\sin(2\pi m/n)$, for all values of $m \leqq n$. In spite of the age of Bessel's formulae, only brief and inadequate tables have been available until recently.

The present work is divided into four parts, an "Introduction," and three tables called respectively "Schedules for Harmonic Analysis and Synthesis," the "Register" and the "Index."

The first table, namely the Schedules, provides 5 decimal values of the two functions given above for values of n from 3 to 100 and for values of $m \leqq n$. With respect to his choice of 5D the author says that the functions "are given to five decimal places in spite of the well-known fact that three decimal places are sufficient for most geophysical investigations. All the values were computed to seven decimal places and I thought it desirable to retain five for those who require greater accuracy, especially as the presence of the additional figures is no handicap to the normal user of the Schedules." The author of this review applauds this decision. He has had occasion to prepare a similar set of tables to $n = 75$ wherein 8 decimal places are retained. The needs of the astronomer, for example, require much greater accuracy than the needs of the geophysicist, the meteorologist, and the economist who are frequent users of these methods.

The Schedules occupy 56 pages of the work, but from the repetitive character of the values tabulated the tables could have been given in half the space. The present generous use of space was adopted, however, because after each value there is given an "identification number" which permits the determination of the corresponding angles by reference to the Index.

The Register provides us, in order of size, the values of the angles $2\pi m/n$, which are given in degrees, minutes and seconds to hundredths of a second, and in radians to 6D. Identification numbers are also provided which by reference to the Index permit the determination of the corresponding values of the sine and cosine.

The Index gives the values of the angles $2\pi m/n$ in terms of the identification numbers. These values are identical with those printed in the Register, but are arranged differently. The corresponding value of the sine is recorded to 5D for each angle together with the value of m and n which determines the angle.

In addition to these principal tables the work gives 12 auxiliary tables used in connection with illustrative examples. A bibliography is appended which contains 17 references. This is quite inadequate and omits reference to such modern works as those of A. HUSSMANN (1938), P. TEREBESI (1930), and K. STUMPFF (1939). See, for example, *MTAC*, v. 1, p. 193, and v. 2, p. 32.

The sines and cosines were computed twice, once by the author and again by C. HEILFRON, the latter using E. GIFFORD, *Tables of the Natural Sines* (see *MTAC*, v. 1, p. 24f). Also about ten per cent. of the values selected at random were checked by F. E. DIXON using Chambers' *Mathematical Tables* (edited by J. PRYDE).

The book is well printed and should be very easy to use in the applications for which it is designed.

H. T. D.

417[K, M].—NIELS ARLEY, *On the Distribution of Relative Errors from a Normal Population of Errors*. Danske Videnskab. Selskab, *Math.-fysiske Meddelelser*, v. 18, no. 3, 1940, 62 p. 14.8 × 24.2 cm.

On p. 61 is a table of r in the formula

$$P(r) = 2\int_r^{(f+1)^{\frac{1}{2}}} \pi^{-\frac{1}{2}}(f+1)^{-\frac{1}{2}}[\tfrac{1}{2}(f-1)]!\left(1 - \frac{r^2}{f+1}\right)^{\frac{1}{2}(f-2)} dr/[\tfrac{1}{2}(f-2)]!$$

for $f = 1(1)30(5)50(10)100, 120, \infty$, $P = [.001, .01, .02, .05, .1(.1).9; 3D]$. There is also a table of $(f+1)^{\frac{1}{2}}$ to 4D.

418[L].—HARVARD COMPUTATION LABORATORY, *Six-Place Tables of* (*a*) $J_m(x)$, (*b*) $Y_m(x)$, (*c*) $J_m(x) - J_{m-2}(x)$, (*d*) $Y_m(x) - Y_{m-2}(x)$, (*e*) $(1/i^m)H_m^{(1)}(x)$, (*f*) $(1/i^{m-1})dH_m^{(1)}(x)/dx$. *Publication no. 21*, July 1945. v, 11 leaves. 21.5 × 27.8 cm. Out of print.

These tables are only for $x = 24, 28, 32, 36, 40$ (a), (b), $m = 0(1)x$; (c), (d), $m = 0(1)x + 1$; and (e), (*f*), real and imaginary parts, $m = 0(1)x$.

419[L].—NBSMTP, ***Tables of Spherical Bessel Functions*****, volume 1, New York, Columbia University Press, 1947. xxviii, 375 p. 20 × 26.4 cm. $7.50.**

Considering that of the eleven coordinate systems for which the wave equation is separable, the solutions in six involve Bessel functions, and that in four the orders are half odd integers, it is—as Professor PHILIP M. MORSE remarks in the foreword to this volume—surprising that extensive previous tables of these functions do not exist (see FMR, *Index*, §17.21, p. 248, or *MTAC* v. 1, p. 234). The surprise is not lessened by the fact that one of these coordinate systems is the obviously important spherical polar, nor by the fact that the functions are often linked with the name of Stokes.

The combination to which the analysis of the wave equation (and others) gives rise is $x^{-\frac{1}{2}}J_{n+\frac{1}{2}}(x)$ rather than the function $J_{n+\frac{1}{2}}(x)$ itself, and it is a numerical multiple of the former function which is tabulated in the main table in this volume, namely $(\pi/2x)^{\frac{1}{2}}J_{\pm(n+\frac{1}{2})}(x)$.

When we note that

$$(\pi/2x)^{\frac{1}{2}}J_{\frac{1}{2}}(x) = \sin x/x,$$

$$(\pi/2x)^{\frac{1}{2}}J_{-\frac{1}{2}}(x) = \cos x/x,$$

and that the functions of higher order are expressible in the form

$$P_n \cos x + Q_n \sin x,$$

where P_n and Q_n are polynomials in $1/x$ (i.e., that the 'asymptotic' expansions terminate in this case) *some* reason—but still hardly adequate—can be discerned for the delay in tabulating this class of function.

The main table in this volume (p. 2–323) is one of

$$(\pi/2x)^{\frac{1}{2}}J_{\pm(n+\frac{1}{2})}(x)$$

for $n = 0(1)13$, and $x = 0(.01)10(.1)25$, expanded for $n = 13$ in the range $x = 10(.05)10.5$. The standard of accuracy may be loosely described as 8S (at least) for $x < 10$ and 7S for $x > 10$. Actually, for smaller values of x and for the smaller positive orders, there are large ranges where as many as 10S are given. For values of x where the functions have begun to oscillate, it would be more accurate to say that enough *decimals* are given for the above standards to apply to the *maxima*. Functions of equal and opposite order are tabulated together, each pair occupying 23 pages.

Interpolation is provided for by second (or modified second) differences, where these are adequate, and a table (p. 370–375) of the Everett coefficients is given. But for the smaller values of x, and especially for the higher orders, the functions vary so rapidly that (even modified) second differences become inadequate, and, in fact, interpolation by differences becomes not feasible. Hence there are many empty columns headed δ^2 (sometimes when the companion function has δ^2_m provided). The need for interpolation in these ranges is met by the provision of auxiliary tables (p. 326–369) giving $(\pi/2)^{\frac{1}{2}}\cdot x^{-\nu}J_\nu(x)$, with δ^2. (In this part of the table we encounter many columns headed δ^4, entirely blank!)

These tables should prove of very great value. They will enable the solutions of many physical and engineering problems to be explored in considerable numerical detail, where previously the labour would have been quite impracticable for the physicists or engineers concerned. The account of the methods of computation and checking, coupled with the reputation for accuracy which the NBSMTP has earned, enables one to use these tables with complete confidence.

In view of the undoubted value of the tables, and of the amount of useful material they contain, it may seem ungenerous to find fault. But, in the end, nothing but the best is good enough! In some respects the design and arrangement of these tables falls short of the standards which the NBSMTP has itself set, and maintained in its earlier volumes. The number of empty *headed* columns has already been mentioned. Another point is inconsistency in breaking up the entries into groups by spaces. In most of the tables the now usual (and for *many* reasons optimum) groups of five, with breaks every five places from the decimal point, are used, but in places groups of six or more are given. The arrangement of p. 321 is not

happy, and there seems no obvious reason why additional horizontal spaces could not have been provided on this (exceptional) page. Attention to such detail would have required an infinitesimal proportional increase in the time and labour which has been expended on this volume.

W. G. BICKLEY

EDITORIAL NOTE: The second volume of NBSMTP, *Spherical Bessel Functions* is to contain the following:

(a) Tables of $(\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu(x)$, for $\pm 2\nu = 29(2)43$, $x = 0(.01)10(.1)25$, and $\pm 2\nu = 45(2)61$, for $x = 10(.1)25$.
(b) Tables of $\Lambda_\nu(x) = 2^\nu\Gamma(\nu + 1)J_\nu(x)/x^\nu$, for $x = 0(.1)10$, 2ν = [1(1)41(2)61; 8–9S], and $x = 10(.1)25$, 2ν = [1(2)61; 7S mostly]. Also for negative values of ν, in regions where $(\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu(x)$ does not difference well.
(c) Table of roots of $J_\nu(x)$ and $J_\nu'(x)$ over the region covered by the main tables.
(d) Other auxiliary tables for purposes of interpolation.

The publication of this volume is expected by September 1947.

420[L].—P. M. WOODWARD & Mrs. A. M. WOODWARD, with the assistance of Miss R. HENSMAN, H. DAVIES, & Miss N. GAMBLE, "Four-figure tables of the Airy function in the complex plane," *Phil. Mag.*, s. 7, v. 37, Apr. 1946, publ. Jan. 1947, p. 236–261. 17 × 25.3 cm. See RMT **260**, *MTAC*, v. 2, p. 35.

"Prefatory Remarks: The immediate need for tables of the Airy function in the complex plane has arisen in connection with theoretical work on the propagation of electromagnetic waves through the earth's atmosphere, and it was for this particular purpose that the present tables were computed and produced as a report at the Telecommunications Research Establishment in February, 1945. It is not claimed that the tables are comprehensive, but they do provide material which, so far as is known at present, is not available elsewhere. The introductory matter, moreover, should be of interest independently of the tables, as it provides general suggestions with regard to interpolation in functions of a complex variable. Much of the labour associated with ordinary second and fourth difference bivariate interpolation may be avoided if suitable use be made of the Cauchy-Riemann equations and if the differences tabulated be modified accordingly."

The functions Ai(z) and Bi(z), $z = x + iy$, are independent solutions of $d^2w/dz^2 = zw$, such that $\mathrm{Ai}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{Bi}(z) = 1/\pi$. The power series

$$w_1 = 1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!}z^6 + \frac{1\cdot 4\cdot 7}{9!}z^9 + \cdots,$$

$$w_2 = z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \frac{2\cdot 5\cdot 8}{10!}z^{10} + \cdots$$

were used for calculation of the functions by means of the relations $\mathrm{Ai}(z) = pw_1 - qw_2$, $\mathrm{Bi}(z) = 3^{\frac{1}{2}}(pw_1 + qw_2)$, where $1/p = (-1/3)!\,3^{\frac{1}{3}}$, $1/q = (-2/3)!\,3^{\frac{1}{3}}$.

There are 8 tables containing the real and imaginary parts of I–IV: Ai(z), Bi(z); V–VIII: Ai′(z), Bi′(z). These cover the region $x = -2.4(.2) + 2.4$, $-y$ = [0(.2)2.4; 4D] in the lower half of the complex plane. The upper half is covered by taking the complex conjugates from the tables. Beneath each tabular entry are given $\frac{1}{2}(\Delta_x^2 - \Delta_y^2)$ and $\Delta_x^2\Delta_y^2$. It is hoped that, apart from any typographical errors which may have passed undetected, the errors in Tables I–IV never exceed 0.6 or 0.7 and those of Tables V–VIII never exceed 0.8 in units of the fourth decimal.

Postscript: Since the preparation of these tables, there have been published ["Annals of the Computation Laboratory of Harvard University," v. 2, ··· 1945] eight-figure tables of the modified Hankel functions of order one-third[1]—functions which are closely related to the Airy functions. The tables cover the region $|x + iy| \leqslant 6$, at interval 0.1 in x and y.

Extracts from the text

[1] See *MTAC*, v. 2, p. 176f.—EDITORIAL NOTE.

421[L, M].—J. P. KINZER & I. G. WILSON, "End plate and side wall currents in circular cylinder cavity resonator," *Bell System Technical Jn.*, v. 26, Jan. 1947, "Appendix, integration of $\int_0^x J_n(x)dx/J_n'(x)$," p. 70–79; tables calculated by Miss F. C. LARKEY. 15 × 22.8 cm.

On p. 73–75 are 4D tables of $F_n(x) = \int_0^x J_n(t)dt/J_n'(t)$, and of $G_n(x) = e^{-F_n}$, for $x = 0(.1)9.9$, $n = 1(1)3$. Then on p. 76–79 are 4D tables of $J_n(x)$ for $x = 0(.1)9.9$, $n = 0(1)7$, and of $J_n'(x)$, for $x = 0(.1)9.9$, $n = 1(1)6$.

422[L, P].—V. I. KOVALENKOV, "Obshchee reshenie uravneniia Gel'mgol'tsa s uchetom vliianiia zheleza" [A general solution of the Helmholtz equation taking into account the effect of iron], Akad. N., Moscow, *Otdiel Tekhnicheskikh Nauk, Abtomatika i Telemekhanika*, Organ komiteta Telemekhaniki i Abtomatiki, no. 2, 1939. 16.5 × 25.1 cm.

On p. 9 there is a table of values of

$$\frac{y}{1!} + \frac{y^2}{2(2!)} + \frac{y^3}{3(3!)} + \cdots, \tag{1}$$

for $y = 0(.01)5$, 4D through 3.25, 3D or 5S thereafter.

On p. 22 is a table of values of

$$\frac{y}{1!} - \frac{y^2}{2(2!)} + \frac{y^3}{3(3!)} - \frac{y^4}{4(4!)} + \cdots \tag{2}$$

for the same range, 4D through 1.34, 3D or 4S thereafter.

423[L, S].—GUY LANSRAUX, "Calcul des figures de diffraction des pupilles de révolution," *Revue d'Optique*, v. 26, Jan.–Feb. 1947, p. 24–45. 15.6 × 23.9 cm.

On p. 43–45 is a table of $\Lambda_n(x) = L_n(x) = n!\,(\tfrac{1}{2}x)^{-n}J_n(x)$, for $n = 1(1)30$, and $x = [0(.5)5(1)15; 6D]$.

If $G(x) = e^{-i\theta}[L_1(x) + \tfrac{1}{2}i\theta L_2(x) + \cdots (1/n!)(i\theta)^{n-1}L_n(x) + \cdots]$,
$G_1(x) = \cos\theta[L_1(x) - (\theta^2/6)L_3(x) + \cdots] + \sin\theta[\tfrac{1}{2}\theta L_2(x) - (\theta^3/24)L_4(x) + \cdots]$,
$G_2(x) = \cos\theta[\tfrac{1}{2}\theta L_2(x) - (\theta^3/24)L_4(x) + \cdots] - \sin\theta[L_1(x) - (\theta^2/6)L_3(x) + \cdots]$,
and $I(x) = G_1^2(x) + G_2^2(x)$. On p. 32 are tables $G_1(x)$, $G_2(x)$ to 5D, and of $I(x)$ to 4D, for $\theta = \pm\pi, \pm 2\pi$, $x = 0(.5)5(1)15$; there are also graphs of the six functions on p. 32–33.

There are also tables on p. 37 of:

$G(x) = e^{-k}[L_1(x) + \tfrac{1}{2}kL_2(x) + \cdots + (k^{n-1}/n!)L_n(x) + \cdots]$, for $k = 0$, to 5D, and for $k = 1, 2, 4$, to 4D; on p. 36 are graphs of the four functions.

On p. 40 are tables of:

$$G_1(x) = \tfrac{1}{2}(\tfrac{1}{2}\pi)L_3(x) - \frac{1}{7\cdot 3!}(\tfrac{1}{2}\pi)^3L_7(x) + \cdots,$$

$G_2(x) = -L_1(x) + \dfrac{1}{5\cdot 2!}(\tfrac{1}{2}\pi)^2L_5(x) + \cdots$, and $I(x) = G_1^2(x) + G_2^2(x)$, for $x = 0(.5)5(1)15$, the first two to 5D and the third to 4D. On p. 41 are graphs of the functions.

In JAHNKE & EMDE, *Tables of Functions*, 1945, p. 180–188, there are tables of $L_n(x)$, for $n = 0(1)8$, $x = [0(.02)9.98; 4\text{–}5D]$. But the NBSMTP tables of $L_n(x)$, see *MTAC*, v. 1, p. 363f, are for $n = 0(1)20$, $x = [0(.1)25; 10D]$; $n = 0(1)12$, $x = [0(.01)10; 8D]$. Thus the Lansraux tables of $L_n(x)$ are new only for $n = 21(1)30$, $x = [0(.5)5(1)15; 6D]$.

R. C. A.

424[M].—NIELS ARLEY, *On the Theory of Stochastic Processes and their Application to the Theory of Cosmic Radiation*. Diss. Copenhagen G.E.C. Gads Forlag, 1943. 240 p. 17.3 × 25 cm.

Chapter 8, p. 222–227 is entitled, "On the numerical computation of $\psi(x) = \int_0^x e^t dx$," $t = x^2$, that is, Dawson's or Poisson's integral, tables of which we have listed, *MTAC*, v. 1, p. 322–323, 422–423, v. 2, p. 55, 185. Arley's only references are to JAHNKE & EMDE, and DAWSON. T. 24, p. 224–225, gives $\psi(x)$, for $x = [2(.01)10; 4S]$. T. 23, p. 223, is of $f(x)$ $= 2xe^{-t}\psi(x) = 1 + \dfrac{1}{2x^2} + \dfrac{1\cdot3}{(2x^2)^2} + \dfrac{1\cdot3\cdot5}{(2x^2)^3} + \cdots$, $x \gg 1$, for $x = [2(.01)4(.1)10;\ 4\text{–}5S]$, $t = x^2$. Of this table it is stated that "We estimate that the figures are correct within 1 or 2 units in the last figure." T. 24 was calculated from T. 23 by means of the formula $\psi(x) = (2x)^{-1}e^t f(x)$, $t = x^2$.

R. C. A.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT 412 (Loria); N **74** (FMR).

105.—G. F. BECKER & C. E. VAN ORSTRAND, *Hyperbolic Functions* (*Smithsonian Mathematical Tables*), Washington, 1909—fourth reprint 1931.

In this and previous editions the formulae 83, 84 on p. XV are incorrect. They should read

$$83.\quad \tanh^{-1}\tan u = \tfrac{1}{2}gd^{-1}2u$$
$$84.\quad \tan^{-1}\tanh u = \tfrac{1}{2}gd2u$$

They are given correctly in the fifth reprint 1942. The same error occurs in A. E. KENNELLY, *Tables of Complex Hyperbolic and Circular Functions*, second ed., Cambridge, Mass., 1921, p. (230), and in various textbooks.

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106. W. W. DUFFIELD, *Logarithms, their Nature, Computation, and Uses, with Logarithmic Tables of Numbers and Circular Functions to Ten Places of Decimals*. Washington, 1897. See *MTAC*, v. 2, p. 161–165.

A. The fact that Duffield prepared this table by copying from VEGA *Thesaurus*, 1794, is well known; see the references in *MTAC* (above), and also L. J. COMRIE, Br. Astron. Assoc., *Jn*., v. 36, 1926, p. 341.

DUFFIELD's copying from Vega included nearly 300 end-figure errors, but his work is held in such low esteem that no one appears to have checked anything but the end figures. Recently an opportunity of examining the first six presented itself when reading the proofs of a new 6-figure table now in press for Messrs. W. and R. Chambers. The result of a single comparison was as follows.

Page	Number	
495	32067	the figures 058 3318 should be overlined
500	33411	*for* 8994744, *read* 8894744
526	41764	*for* 8920871, *read* 8020871
530	42680	*for* 2224108, *read* 2244108
580	57482	*for* 3318704, *read* 5318704
686	89330	the overline on 9973340 should be deleted
710	9680.	in number, *for* 7180, *read* 9680
710	96826	the overline on 9919910 should be deleted
716	98771	*for* 5294508, *read* 6294508.

In *Science*, n.s., v. 7, 1898, p. 109–111, there is a devastating editorial on this table entitled "Logarithms on the 'Spoils System.'" It is unsigned, but from the fearless vigour of its style, and the fact that SIMON NEWCOMB was the mathematics member of the editorial committee,[1] there can be little doubt of the authorship. A few quotations, even after a lapse of nearly half a century, may not be without interest. "Anybody who knows anything about the subject knows that useful tables of logarithms include from four to seven places. The number of problems in which a table of more than seven places would be used is extremely small, and all extension of figures over what are actually used are a nuisance and a real hindrance. That the United States government should suddenly print for free distribution several thousands of copies of this compilation must create, among those who understand, a strong suspicion of a dearth of other printable material." Then, with biting sarcasm: "Their arrangement might have been worse, but only by printing the numbers in one annual report and their logarithms in the next." Of Duffield's claim that he did not know of Vega when the computations were begun: "This great work of Vega, which every tyro in computing knows, was published in 1794. This is more than a hundred years ago, and it is not easy to understand how one could seriously think of repeating such a performance without finding that it had already been done." Then, with his tongue in his cheek: "The author thinks he has discovered some serious mistakes in Vega, but he delicately refrains from telling what they are."

L. J. C.

[1] Simon Newcomb was mathematical editor of *Science* 1895–1903. EDITOR.

B. Since L. J. C. had many years ago sent to me a copy of his 1926 review of Henderson's *Bibliotheca Tabularum Mathematicarum*, 1926, to which he refers above, it was a decided oversight on my part to omit a reference to it in RMT **319**. The paragraph of the review which is here pertinent is the following: "However, he did not take Duffield at his face value, and so made the discovery of his dishonesty in attempting to pass a copy of Vega as his own computation. Peters was also aware [1922] of this fraud, and it was noticed independently by the present writer in 1924. Henderson's hypothesis that Duffield was original up to 26000 is untenable. It is far more probable that in the copy of Vega's *Thesaurus* which Duffield used some previous owner had entered the corrections given by Lefort up to 26000. This supposition is supported by the fact that of the five errors in Duffield before 26000 four were not given by Lefort; three were first pointed out by Glaisher, and in one case Lefort had omitted the asterisk which denoted that the error was in Vega as well as in Vlacq." Before commenting on this I should point out that I did not with sufficient clearness indicate that of the two errata lists of Lefort the first, of 1858, was a list of errors in Vlacq, *Arithmetica Logarithmetica*, 1628, upon which Vega's table of logarithms was based; and the second, of 1875, was simply a list of errors in Vega's *Thesaurus*. In the first list, however, Lefort added a star to indicate where an error in Vlacq persisted in Vega.

L. J. C. kindly reported to me that the "five errors" in Duffield before 26000, to which he referred, were in connection with the numbers 10033 (Lefort with * omitted), 11275 (Glaisher, R.A.S. *Mo. Not.*, v. 32, p. 258), 11699 (Glaisher, *idem*, v. 32, p. 258), 22312 (Lefort), 24580 (Glaisher, *idem*, v. 32, p. 258 and v. 34, p. 471). It is true that Lefort 1858 does not list an error in Vega in connection with 10033 but Lefort 1875 *does* list such an error. Thus in the five Duffield errata two were listed by Lefort. With regard to the error associated with 11275 Glaisher remarks the logarithm of 11275 is 4.05211, 65505, 49998, 14..., and it is a matter of indifference whether the tenth figure of the mantissa be increased or not. But Glaisher lists the end figures of Duffield 65506 as an "error" and the "correction" 65505. That this error was also in Peters was pointed out by L. J. C. in MTE **104**. Thus Peters listed only four Duffield errata before 26000, not five.

R. C. A.

107. MAURICE KRAÏTCHIK, *Recherches sur la Théorie des Nombres*, v. 1. Paris, 1924.

On p. 77–80 is a table of the factors of the two Fibonacci sequences

$$1, 1, 2, 3, 5, 8, 13, \cdots, U_n, \quad \text{and} \quad 1, 3, 4, 7, 11, 18, \cdots, V_n.$$

The following two errata may be noted:

$$\textit{for } U_{37} = 79\cdot 149\cdot 2221, \textit{ read } 73\cdot 149\cdot 2221,$$
$$\textit{for } U_{67} = 44945570212853, \textit{ read } 269\cdot 116849\cdot 1429913.$$

This latter error appears to be due to POULET, since the entry is attributed to him.

Table I, p. 131–191 gives, in effect, for each prime less than 300000 the exponent e of 2 modulo p, that is the smallest e for which $2^e - 1$ is divisible by p. Actually, to save space, the table gives $\gamma = (p - 1)/e$. This table is the most extensive of its kind and has been used to a considerable extent in connection with tables of factors of $2^n \pm 1$ and other problems involving the binomial congruence. Immediately after its publication this table was compared with a set of similar tables of CUNNINGHAM & WOODALL[1] extending to $p < 100000$ and the resulting errata of 44 items appear in *Messenger Math.*, v. 54, 1924, p. 184 (given also in *Guide to Tables in the Theory of Numbers*. Washington, 1941, p. 155.)

The purposes for which this table is most frequently used require information about primes whose exponents are comparatively small. It therefore seemed desirable to find independently those primes whose exponents do not exceed 2000 in the range $100000 < p < 300000$. A comparison of these results with Kraitchik's table yields most of the entries in the errata list given below.[2] Since this comparison involves less than 1.5% of the entries in Kraitchik's table the user of this table, who is interested in exponents beyond 2000, is exposed to considerable risk.

p	*For*	*Read*	p	*For*	*Read*
[2]104161	60	30	[4]165313	96	672
101737	4	8	165233	4	92
102043	2	9	194867	7	217
106649	8	4	216217	24	168
107857	7	14	220243	3	213
108497	8	16	246739	3	177
108967	39	78	247381	1	217
[3]109121	8	248	247531	5	185
111487	6	102	250867	2	1
[2]114601	2	6	254039	2	142
119929	2	114	255071	1	2
[2]121081	4	20	[2]267481	1	2
[2]127681	8	152	272959	2	938
141023	98	14	[5]284689	2	216

TIETZE notes[6] that on p. 191, after 297 967, for 297 671, should be 297 971. For other misprints of primes see my *Guide*, p. [156].

D. H. L.

[1] A. J. C. CUNNINGHAM & H. J. WOODALL "Haupt-exponents of 2," *Quart. Jn. Math.* v. 37, 1905, p. 122–145; v. 42, 1911, p. 241–250; v. 44, p. 41–48, 1912, p. 237–242, 1913; v. 45, 1914, p. 114–125.

[2] Dr. A. E. WESTERN has already noted five of these in *MTAC*, v. 1, p. 429.

[3] This error caused the omission of the entry: 96135601, 881 in the writer's table of composite solutions of $2^n \equiv 2 \pmod{n}$, *Amer. Math. Mo.*, v. 43, 1936, p. 351; see MTE **102**.

[4] The discovery of this error leads to the following factorization into primes

$$2^{133} + 1 = 3^2\cdot 83\cdot 739\cdot 165313\cdot 8831418697\cdot 13194317913029593$$

[5] Kraitchik had here $284687 = 13\cdot 61\cdot 359$, which is therefore not a prime; the value of γ was also incorrect.

[6] Akad. d. Wissen., Munich, *Abh.*, n.s. Heft 55, 1944, p. 9; see RMT **369**.

108. NBSMTP, *Tables of Sine, Cosine and Exponential Integrals*, v. 1, 1940.

P. 59, argument column, *for* 1.1405, *read* 0.1405.

109. NBSMTP, *Tables of the Exponential Function* e^x, 1939. See *MTAC*, v. 1, p. 438.

P. 168, x = 1.6742, *for* 5.33452 58202 12879, *read* 5.33452 58209 12879.
P. 304, x = .2333, *for* .79181, *read* .79191.

UNPUBLISHED MATHEMATICAL TABLES

56[B].—Great Britain, Admiralty Computing Service, *Tables of* $x^{1/4}$, $x^{-1/4}$, $x^{3/4}$, $x^{-3/4}$. Machine printed copy prepared by and in the possession of H. M. Nautical Almanac Office. Compare RMT **339**, *MTAC*, v. 2, p. 205.

Several requirements arose for quarter powers during the course of the computational work undertaken by Admiralty Computing Service at H. M. Nautical Almanac Office during 1943–1945. In the same period the Office was faced with the training of new staff with no previous computing experience. It was accordingly decided to make systematic tables of the four powers $-\frac{3}{4}, -\frac{1}{4}, +\frac{1}{4}, +\frac{3}{4}$ for a comprehensive range of argument; by this means considerable individual calculation for special investigations was avoided and the new staff provided with excellent material for elementary training in computing and tabulation.

Copy has been prepared for two tables in both of which the four functions are arranged side by side in the order $+\frac{1}{4}, -\frac{1}{4}, +\frac{3}{4}, -\frac{3}{4}$ for the range x = 1(.01)10(.1)100(1)1000(10)10000.

Table A. An accurate table with at least 7S with manuscript first differences written in small figures interlinearly. The number of decimals retained is:

Range	Power			
x	$+\frac{1}{4}$	$-\frac{1}{4}$	$+\frac{3}{4}$	$-\frac{3}{4}$
1–10	6	7	6	7
10–100	6	7	6	8
100–1000	6	7	5	9
1000–10000	6	7	4	9

Table B. A "working" table to 11 or 12S intended solely to give the tabulated values to the greatest accuracy to which they are available; therefore no differences are provided and the end-figure may be in error by several units. The number of decimals (D) and the error (E) in the last figure which is unlikely to be exceeded are given in the following table:

Range	Power							
x	$+\frac{1}{4}$		$-\frac{1}{4}$		$+\frac{3}{4}$		$-\frac{3}{4}$	
	D	E	D	E	D	E	D	E
1–10	10	2	10	2	10	2	10	2
10–100	10	3	10	2	9	2	11	4
100–1000	9	2	10	2	8	2	12	5
1000–10000	9	2	10	2	7	2	12	2

The original aim was to provide a table giving 7S accuracy throughout, interpolable with only trivial second difference corrections. Basic values were calculated to 10 or 11S for x = 1(.01)3(.05)6.5(.1)10 for powers $\pm\frac{1}{4}$; x = 1(.01)4(.05)7.5(.1)10 for powers $\pm\frac{3}{4}$. These were multiplied by the appropriate powers of 10 to give powers of $10x$, $100x$ and $1000x$ over the same ranges of x. Values of all 16 functions were then obtained for a uniform interval of .01 in x, over the whole range x = 1 to 10, by standard methods of interpolation to fifths and tenths on the National machines. The copy in each case was prepared by integrating on the National machine from differences produced by end-figure

differencing of the interpolated values. Each table contains 72 foolscap pages, each with 4×50 entries, apart from differences.

JOHN TODD & D. H. SADLER

57[D].—GREAT BRITAIN, Admiralty Computing Service, *Six-figure logarithmic-trigonometrical tables*. Copy prepared by machine printing with ms. proportional parts, and in the possession of H. M. Nautical Almanac Office.

At the request of the Ministry of Supply, Admiralty Computing Service undertook to prepare copy for a six-figure table of logarithmic-trigonometric functions for use in the optical industry. The argument was to be in degrees and decimals. The obvious course of reproducing by photo-lithography PETERS, *Sechsstellige Logarithmen der trigonometrischen Funktionen von 0° bis 90° für jedes Tausendstel des Grades*, Verlag der Preussischen Landesaufnahme, Berlin 1921, was ruled out as being too expensive; it was decided to compromise by photographing the first five degrees of Peters' table and adding to it a new table at interval 0°.01.

It is this latter table, giving logarithms of the four functions sine, tangent, cotangent and cosine in the range 5°(0°.01)45°, which has been prepared. For various reasons, it was later decided not to proceed with the publication of the table.

The copy was prepared by building up from the second differences of the known end-figures; it was prepared on the National machine in a form as closely similar to Peters' table as the limitations of the machine would permit. Proportional parts were written in by hand. The copy of 80 p. is in perfect condition for the printer, but not good enough for photo-lithography.

JOHN TODD & D. H. SADLER

58[D, E].—GREAT BRITAIN, Admiralty Computing Service, *Tables of* $\operatorname{cosec}^2 x - x^{-2}$ *and* $x^{-2} - \operatorname{cosech}^2 x$. On National differencing sheets and in MSS. prepared by and in the possession of H. M. Nautical Almanac Office.

In connection with a special investigation tables were required of the two functions

$$\operatorname{cosec}^2 x - x^{-2} \quad \text{and} \quad x^{-2} - \operatorname{cosech}^2 x$$

to 14D for the range $x = 0(.001)1.6$. They have been calculated to 17D for $x = 0(.01)1.69$ and now await interpolation to tenths.

The fundamental values have been deduced from the tables of C. E. VAN ORSTRAND, Nat. Acad. Sci., *Memoirs*, v. 14, no. 5, 1921.

JOHN TODD & D. H. SADLER

59[L].—GREAT BRITAIN, Admiralty Computing Service, *Tables of* $C(m, \mu; x)$. Machine printed copy prepared by, and in the possession of H. M. Nautical Almanac Office.

These tables of the hypergeometric function

$$C(m, \mu; x) = F(1, m - \mu; m, x)$$

have been prepared at the suggestion of Dr. W. G. BICKLEY and Dr. J. C. P. MILLER, with a view to their use in the summation of certain slowly convergent series. Consider a series Σu_n, in which the ratio of consecutive terms can be written, to a good approximation for values of n greater than N, in the form:

$$u_{n+1}/u_n = x(1 - \mu n^{-1} + \mu\nu n^{-2})$$

where μ and ν are constants.

Then, to the same approximation,

$$u_{n+1}/u_n = x(m - \mu)/m, \qquad \text{where} \qquad m = n + \nu,$$

and so

$$\sum_{\nu=0}^{\infty} u_{N+\nu} = u_N C(m, \mu; x).$$

A good approximation to the tail of the series is thus obtained.

The function is tabulated to 4D for $4m = 40(1)44, 80(1)84$; $4\mu = 4(1)20$; $x = .8(.02).9(.01)1$.

No differences are given, since it is assumed that the table will generally be used without interpolation. One page is devoted to each value of x and contains the two double-entry tables corresponding to the two series of values of m.

The method of calculation *should* result in the last figures not being in error by more than .6, but the system of checking adopted was not capable, throughout the whole range, of guaranteeing an accuracy of more than one unit.

The general method of computation was the repeated application of the recurrence formulae

$$C(m + 1, \mu; x) = \frac{m}{(m - \mu)x}\{C(m, \mu; x) - 1\}$$

and

$$C(m, \mu; x) = (\mu - 1)^{-1}\{(m - 1) - (m - \mu)(1 - x)C(m, \mu - 1; x)\}$$

due regard being paid to the loss of figures inherent in this method. Initial values were obtained from the formulae

$$C(1, \mu; x) = (1 - x)^{\mu-1}; \mu \neq 1,$$

and

$$C(2, 1; x) = -x^{-1} \ln (1 - x)$$

with

$$C(m, \mu; 1) = (m - 1)/(\mu - 1).$$

JOHN TODD & D. H. SADLER

MECHANICAL AIDS TO COMPUTATION

In the introductory article, "Admiralty Computing Service," there are reviews of publications **37**, *Electronic Differential Analyser;* **40**, *Rangefinder Performance Computer;* **112**, *The Fourier Tranformer.*

There is an interesting biographical sketch, and a portrait of HOWARD HATHAWAY AIKEN (1900–), in *Current Biography*, v. 8, no. 3, March 1947.

In *Wisconsin Engineer*, v. 51, Dec. 1946, p. 10–12 is an article by WALTER GRAHAM, "Do you know your slide rule?", in which the author explains the slide rule solution of equations of the type $x^x = k$, $a^x = x^b$, $\tan x = kx$, and $\sin x = kx$. See *MTAC*, v. 1, p. 203, Q **8** and v. 2, p. 194, **25**.

Dr. LOTHAR SCHRUTKA, professor of mathematics in the Technische Hochschule, Vienna, is the author of a third much revised edition of his *Theorie und Praxis des logarithmischen Rechenschiebers*, Vienna, Deuticke, 1943. xii, 101 p. The first edition appeared in 1911, and the second in 1929. There are 32 titles in the bibliography, p. 95–96, and there is a full index, p. 97–101.

H. H. AIKEN & GRACE M. HOPPER, "The Automatic Sequence Controlled Calculator," *Electrical Engineering*, v. 65, Aug.–Nov. 1946, p. 384–391, 449–454, 522–528. See *MTAC*, v. 2, p. 185f.

29[Z].—D. R. Hartree, *Calculating Machines, Recent and Prospective Developments and their Impact on Mathematical Physics.* Cambridge, University Press, 1947. 40 p. + a double plate. 12.4 × 18.1 cm. 2 shillings.

This is a lecture given on the occasion of the author's inauguration as Plummer Professor of Mathematical Physics in the University of Cambridge. Its general purpose was to acquaint his audience with the war time expansion of large scale computing units in the United States. The lecture is divided into 9 parts by the following subheadings: (1) Introduction, (2) Two classes of calculating equipment, (3) Functions of components of a digital machine, (4) The ENIAC, (5) The master programmer, (6) Example of the application of the ENIAC, (7) Prospective developments, (8) The impact of these developments on mathematical physics, (9) Conclusion.

Most of the lecture is devoted to a discussion of the ENIAC, see *MTAC*, v. 2, p. 97–110. Part 6 is a short description of an interesting boundary layer problem which the author put on the ENIAC in June 1946. It consists in solving the non-linear system of three differential equations

$$f' = h(1 + \alpha r)^{-1/\theta}, \qquad h'' = -fh', \qquad \beta r'' = fr' + (h')^2,$$

with the two-point boundary conditions,

$$f = h = r' = 0 \text{ at } x = 0, \qquad h = 2, r = 0 \text{ at } x = \infty.$$

Part 7 is a very brief description of the Electronic Discrete Variable Calculator (EDVAC) type of machine.

Part 8 is an interesting discussion of the way that the possibility of high speed large scale computing alters the outlook of the mathematical physicist. An ordinary system of linear equations becomes a problem in minimizing a quadratic form. A second order partial differential equation with certain boundary conditions becomes an integral equation with "built in" boundary conditions. To quote the author: "The facilities offered by these new calculating machines will at least make the formulating of physical problems in terms of integral equations and variation equations more familiar and may in time wean us from our present tendency to regard a differential equation as the basic way of formulating the mathematics of physical problems."

D. H. L.

30[Z].—Francis J. Murray, *The Theory of Mathematical Machines.* New York, Columbia Univ. Press, 1947. vii, 116 p. 21.5 × 28 cm. Lithographed; plastic binding $3.00.

This work appears to be a set of lectures on certain mathematical aspects of computing devices. There is a wide variety of both devices and aspects. Each of its four parts is divided into four or five short chapters of a few pages each. There are about 200 original line drawings that add much to the interest of the volume. Unfortunately these are not numbered so that sometimes the reader is in doubt as to which drawing is being referred to in the text. The book is lithographed from typewriting and is very neat.

The reviewer has found it difficult to give a short account of the actual contents of the book. There is a great deal of detail (not indicated in the chapter headings) on some topics and very little on others. On the whole the book is devoted to continuous or analogue devices almost entirely; there is only the briefest mention of high speed digital computers. There is a great deal of space devoted to electronics but only one example of an electronic counter, a soft tube prewar type. There is much material of an electro-mechanical nature but no mention is made of the possibility of using relays for computing. The mathematical discussion ranges in depth from the identity

$$4xy = (x + y)^2 - (x - y)^2$$

to Hilbert space. The four parts may be described as follows:

Part I, Digital machines, is disappointingly brief (12 p.) and is divided into counters, adders, multipliers, and "the punch card machine." The discussion is largely devoted to mechanical parts used in desk calculators such as the Leibniz wheel and Napier's bones. There is a description of the Hollerith card sorter.

Part II, Continuous operators, sounds the keynote of the text: Numerical quantities can be represented by physical magnitudes. The magnitudes discussed range from linear displacements to the phase angle of alternating currents. There is a good treatment of linear networks. Among multiplying devices there is a description of "square" gears, variably wound potentiometers, and rectifiers. Integrators and differentiators, both mechanical and electrical, are treated in great detail. The rest of Part II is devoted to the theory of amplifiers, servomechanisms, selsyn units and other electrical devices and their uses in mathematical machines.

Part III, The solution of problems, is devoted to composite machines for solving systems of linear equations, ordinary and partial differential equations. These machines include the network analyzer, differential analyzer and two electronic computers for linear equations. One of the latter has been designed by the author and is fully described. The mathematical treatment here is particularly interesting. The reader will find this material under "Adjusters" (p. 84–94, unfortunately the book has no index).

Part IV, Mathematical Instruments, is concerned with planimeters, integrometers, harmonic analyzers and cinema-integraphs. There is a page and a half of bibliography arranged topically. This does not include a large number of references inserted in the text.

The reader, whether he be interested in mathematical machines from a technical or a purely mathematical point of view, will find something interesting on every page. It is to be hoped that a second volume dealing with the theories of the many other interesting devices developed during the war may be eventually forthcoming.

D. H. L.

NOTES

73. The Checking of Functions Tabulated at Certain Fractional Points.—Many functions involving a parameter ν, in particular Bessel functions $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$, $K_\nu(x)$, etc., besides being tabulated for $\pm$ integral values of ν, as well as for $\nu = 0$, are often given for non-integral values of ν between -1 and 1, especially for $\nu = \pm\frac{3}{4}, \pm\frac{2}{3}, \pm\frac{1}{2}, \pm\frac{1}{3}$ and $\pm\frac{1}{4}$. When it is desired to perform the equivalent of a differencing check upon these or related functions (e.g. the zeros of these functions) considered as a function of ν for fixed x, due to the irregular interval in ν, it is necessary to take the divided differences. For any fixed set of n ν's, it is possible to obtain coefficients of f_ν for the last, i.e. $(n-1)$th, divided difference which should vanish if the function behaves as a polynomial of the $(n-2)$th degree in ν. Thus an error ϵ in any entry f_ν (this includes rounding errors) will usually show up by being multiplied by the coefficient of f_ν.

The coefficients which are given below are for three important cases likely to arise in practice, especially with Bessel functions:

(a) 7th divided difference for f_ν involving the 8 points $\nu = \pm\frac{3}{4}, \pm\frac{2}{3}, \pm\frac{1}{3}, \pm\frac{1}{4}$.
(b) 10th divided difference for f_ν involving the 11 points $\nu = \pm 1, \pm\frac{3}{4}, \pm\frac{2}{3}, \pm\frac{1}{3}, \pm\frac{1}{4}, 0$.
(c) 10th divided difference for f_ν involving the 11 points $\nu = \pm\frac{3}{4}, \pm\frac{2}{3}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, 0$.

In case (c), omitting the $f_{\pm\frac{2}{3}}$ and $f_{\pm\frac{1}{3}}$ leaves 7 points at the uniform interval of $\frac{1}{4}$ in ν, which might be amenable to an ordinary differencing check. It is too cumbersome to work with more than 11 points f_ν. At any rate, if to (c) there were added f_ν for $\nu = \pm 1$, the set of ν's would include ± 1, $\pm \frac{3}{4}$, $\pm \frac{1}{2}$, $\pm \frac{1}{4}$ and 0, which are 9 points, again at the uniform interval of $\frac{1}{4}$, and to which an ordinary differencing check could be applied. The joint use of any two of (a), (b), or (c), when possible, lessens the likelihood of passing a double error which, by compensation, might yield a small divided difference in one case.

Divided Difference Formulae:

(a) $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}$; 7th divided difference

$$= \frac{1}{85085}[21\,28896\,(f_{\frac{3}{4}} - f_{-\frac{3}{4}}) - 42\,45696\,(f_{\frac{2}{3}} - f_{-\frac{2}{3}}) + 174\;49344\,(f_{\frac{1}{3}} - f_{-\frac{1}{3}}) - 183\;30624\,(f_{\frac{1}{4}} - f_{-\frac{1}{4}})].$$

(b) $\nu = \pm 1, \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$; 10th divided difference

$$= \frac{1}{4\;25425}[10\,50192\,(f_{-1} + f_1) - 324\,40320\,(f_{-\frac{3}{4}} + f_{\frac{3}{4}}) + 573\,16896 \times (f_{-\frac{2}{3}} + f_{\frac{2}{3}}) - 2944\;57680\,(f_{-\frac{1}{3}} + f_{\frac{1}{3}}) + 3910\;53312\,(f_{-\frac{1}{4}} + f_{\frac{1}{4}}) - 2450\;44800\,f_0].$$

(c) $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$; 10th divided difference

$$= \frac{1}{4\;25425}[454\;16448\,(f_{-\frac{3}{4}} + f_{\frac{3}{4}}) - 1637\;62560\,(f_{-\frac{2}{3}} + f_{\frac{2}{3}}) + 5376\;98304\,(f_{-\frac{1}{2}} + f_{\frac{1}{2}}) - 18845\;29152\,(f_{-\frac{1}{3}} + f_{\frac{1}{3}}) + 19552\;66560\,(f_{-\frac{1}{4}} + f_{\frac{1}{4}}) - 9801\;79200\,f_0].$$

HERBERT E. SALZER

NBSMTP

74. POISSON'S OR DAWSON'S INTEGRAL AND ANOTHER INTEGRAL.—In W. O. SCHUMANN, *Elektrische Durchbruchfeldstärke von Gasen*, Berlin, 1923, p. 243 and 241, are tables of $f(x) = \int_0^x e^z dt$, 2–5S, $z = t^2$, of $\log f(x)$, 4–5S, and of $e^{-w}f(x)$, 3–6D, $w = x^2$, each for $x = [.1(.1)2.6(.2)7.4]$. The reference to Schumann in FMR *Index*, p. 219, cannot be verified. See *MTAC*, v. 2, p. 55, N**45**; and RMT **378** and **424**, ARLEY. On p. 242 are graphs of $f(x)$, and on p. 234–235 is a small 4D table of $MS(x) = \int_0^x e^{-u}dt$, $u = t^4$, for $x = 0(.1)1, \infty$; graphs of $S(x)$, and of $MS(x)$, are given on p. 237. $M = \Gamma(1.25)$ is the value of the integral when $x = \infty$, and is approximately .9064. More generally BIERENS DE HAAN, *Nouvelles Tables d'Intégrales Définies*. Leyden, 1867, and New York, 1939, Table 26(4), gives $\int_0^\infty e^{-v}dt = n^{-1}\Gamma(n^{-1})$, if $v = t^n$. But $\int_0^x e^{-v}dt = n^{-1}\Gamma(n^{-1}, x^n) = xM(n^{-1},\ 1 + n^{-1},\ -x^n)$ where $\Gamma(p + 1, x) = \int_0^x e^{-x}x^p dx$, or as in K. PEARSON, *Tables of the Incomplete Γ-function*, London, 1922, $y_n = n[\Gamma(1/n)]^{-1} \int_0^x e^{-v}dt = \mathrm{I}(u, p) = I[\sqrt{n}\, x^n, (1 - n)/n]$. For $n = 4$, there is a table, p. 118–126, of y_4 for $u = 2x^4 = [.1(.1)27; 7D]$, $p = -.75$; also, p. 164, $u = [0(.1)6; 5D]$. See also the little tables for this particular case by F. EMDE, *Z. f. angew. Math. u. Mech.*, v. 14, 1934, p. 336–339, $S(x)$, $x = [.8, .9(.01)1, 1.1; 6D]$.

MURLAN S. CORRINGTON & R. C. A.

QUERIES

22. Integral Evaluations.—What methods are available for evaluating the integrals

$$\int_0^t \cos\,(a_0 + a_1x + a_2x^2 + \cdots)dx, \qquad \text{and}$$

$$\int_t^\infty \cos\,(a_1x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)dx/x^n,$$

except for point by point numerical integration?

Murlan S. Corrington

Radio Corporation of America
Camden, N. J.

QUERIES—REPLIES

29. Tables of $N^{3/2}$ (Q**5**, v. 1, p. 131; QR**8**, p. 204, **11**, p. 336, **13**, p. 375, **14**, p. 407).—In L. Potin, *Formules et Tables Numériques*, Paris, 1925, there is a table (p. 416–417) of $N^{3/2}$, $N = \cos\theta$, for $\theta = [0(30')90°; 4D]$.

R. C. A.

CORRIGENDA

As the result of recomputation Dr. J. W. Wrench, Jr. requests that the last three decimal places of each of four values given *MTAC*, v. 1, p. 298, l. 11–12, be corrected to read as follows: ber 15,– 535; bei 15,– 887; ber′ 15,– 317; bei′ 15,– 368.

V. I, p. 33, 47,2, *for* 1778, *read* 1777. P. 468, for Everett, *read* Everett, J. D.

V. 2, p. 65, *for* Block, *read* Bloch; *for* Brendle, *read* Brendel.

P. 250, RMT **362**, col. "Diff. for 1″," the second, fourth, and fifth entries should respectively read: 63 to 2.9, 0.3 to 1.6, 1.6 to 15.8. These corrections are due to slips made in the editorial office; the ms. of L. J. C. was faultless. P. 271, l. − 3, end of line, *for* ($\frac{1}{8}$), 1; *read* ($\frac{1}{8}$)1;. P. 277, l. 6, *for* **54**, *read* **55**; l. 7, *for* **55**, *read* **54**.

Mr. D. F. Ferguson has now (May 23, 1947) carried on his calculation of the value of π to 750D, and discovered errors in Dr. Wrench's computations. The value of π on p. 245 has to be amended 723–743D, and the value of $\tan^{-1} \frac{1}{8}$ on p. 247, 725–743D. No announcement will be made in *MTAC* of the exact corrections here necessary until Mr. Ferguson has completely checked the remaining 58 decimal places of our published value of π.

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
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F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

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K. Statistics
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The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee, or an editor.

Due to the enlargement of *MTAC* and publication of illustrations, **beginning with 1947 the subscription price for each *calendar year* is $4.00, *payable in advance;*** ordinary single numbers $1.25. Earlier ordinary single numbers each $1.00, and all numbers for each of the years 1943 to 1946 inclusive, $3.00. Special single number 7, *Guide to Tables of Bessel Functions*, $1.75, and number 12, $1.50. All payments are to be made to National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers. No sample copies are distributed, but detailed descriptive circulars will be sent upon application.

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